Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

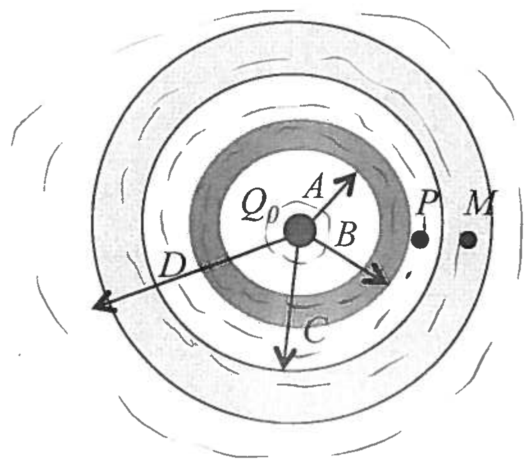
$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \left(i + \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S} \right)$$

Problem 2: (20 points)

An insulating spherical shell of inner radius A and outer radius B has a uniform charge density ρ. It is surrounded by spherical conducting shell of inner radius C and outer radius D. A point charge Q_0 is placed at the center as shown.



a)Find the electric field at

i) $r<A$

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E\,4\pi r^2 = \frac{Q_0}{\varepsilon_0}; \quad \vec{E} = \frac{Q_0}{4\pi\varepsilon_0}\frac{1}{r^2} \text{ rad.out}$$

i) $A<r<B$

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\left(\rho\,\frac{4\pi}{3}(r^3-A^3)+Q_0\right)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(\rho\,\frac{4\pi}{3}(r^3-A^3)+Q_0\right) \text{ rad. out}$$

ii) $B<r<C$

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\left(\rho\,\frac{4}{3}\pi(B^3-A^3)+Q_0\right)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(\rho\left(\frac{4}{3}\pi(B^3-A^3)\right)+Q_0\right) \text{ rad. out}$$

iii) $C<r<D$

$$\vec{E} = 0 \text{ (conductor)}$$

iv) $r>D$

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\left(Q_0+\rho\,\frac{4}{3}\pi\,(B^3-A^3)\right)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(Q_0+\rho\,\frac{4}{3}\pi\,(B^3-A^3)\right) \text{ rad. out}$$

b) Assume that Q_0 is zero. Find the difference in electric potential between point $r = P$ and $r = M$, V(M)-V(P).

$$V(M)-V(P) = -\left[\int_P^C \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(\rho \frac{4}{3}\pi (B^3-A^3)\right) dr + \int_C^M 0 \right] =$$

$$= \frac{1}{4\pi\varepsilon_0}\rho \frac{4}{3}\pi (B^3-A^3)\left[\frac{1}{C}-\frac{1}{P}\right]$$

c) Assume that Q_0 is zero and the insulating shell, inner radius A and outer radius B, has a non-uniform volume charge density $\rho = cr$ where c is a known positive constant. Find the electric field at $B < r < C$.

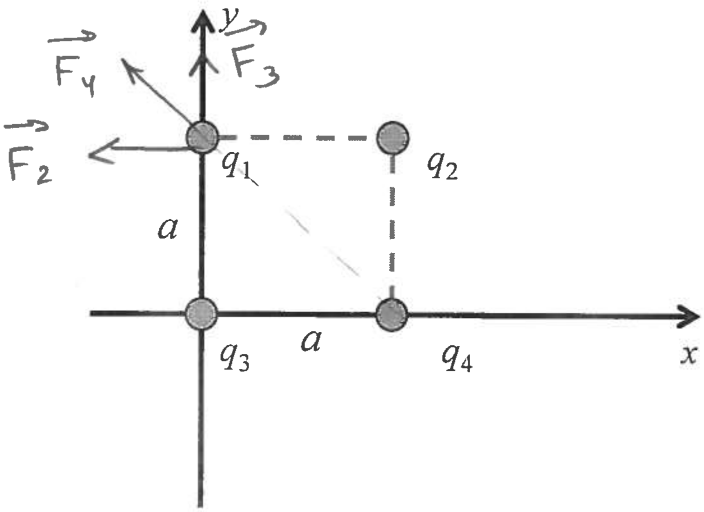
$$E\,4\pi r^2 = \frac{1}{\varepsilon_0}\int_A^B c r\, 4\pi r^2\, dr =$$

$$= \frac{1}{\varepsilon_0} 4\pi c \left.\frac{r^4}{4}\right|_A^B = \frac{1}{\varepsilon_0}\pi c (B^4 - A^4)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\pi c (B^4 - A^4) \text{ rad. out}$$

Problem 3: (15 points)

In the figure below, four particles form a square of side a. The charges q_1 and q_4 are known. What are the charges q_2 and q_3 if the net electrostatic force on particle with charge q_1 is zero?



$$F_{tot_x} = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{a^2} - \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_4}{2a^2}\frac{\sqrt{2}}{2} = 0$$

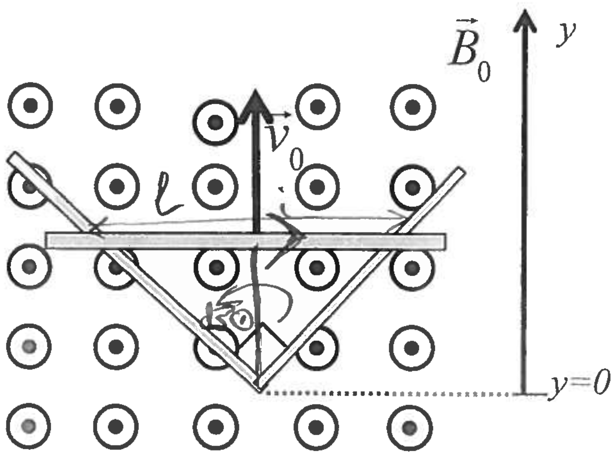
$$F_{tot_y} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{a^2} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_4}{2a^2}\frac{\sqrt{2}}{2} = 0.$$

$$q_2 = -\frac{\sqrt{2}}{4} q_4$$

$$q_3 = -\frac{\sqrt{2}}{4} q_4.$$

Problem 4: (15 points)

Two straight conducting rails form a right angle. A conducting bar with resistance R is in contact with the rails. It starts at the vertex at time t = 0 and moves at constant speed v_0 up. The uniform magnetic field of magnitude B_0 is directed out of the page. [In case you are nervous and forgot the area of a right triangle, it is provided on the last page.]



a) Find the direction of the current in the loop. Explain your answer within this box:

area ↑, Φ ↑, B_{self} ↑↓ B_{ext},

B_{self} ⊗, i is cw

b) Find the current through the bar as a function of time. Ignore self-inductance.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$\Phi = \int \vec{B}\cdot d\vec{S} = B_0 \frac{1}{2}\ell y$$

$$\frac{\ell}{2} = y\tan 45^\circ = y$$

$$\Phi = B_0 y^2$$

$$\frac{d\Phi}{dt} = B_0 2y\frac{dy}{dt} = 2B_0 y v_0 = 2B_0 v_0^2 t$$

$$-iR = -2B_0 v_0^2 t\ ; \quad i = \frac{2B_0 v_0^2 t}{R} \text{ cw}$$

c) Consider the time moment when the bar has length b. What is the magnitude and direction of the force that must be applied to the bar to have the net force zero? Ignore gravity.

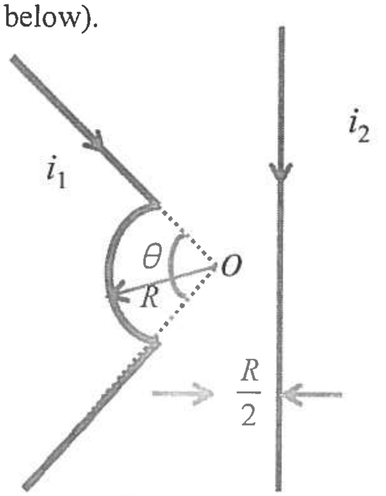
$$d\vec{F} = i\, d\vec{s}\times\vec{B}$$

$$\vec{F}_{mag} = i b B_0 \text{ down}$$

$$\vec{F}_{applied} = i b B_0 \text{ up.}$$

Problem 5: (15 points)

a) There are two wires, each carrying a current. Wire 1 consists of a circular arc of radius R and two radial lengths; it carries current i_1 in the direction indicated. The angle θ is given. Wire 2 is infinitely long and straight; it carries a current i_2 as shown. It is at distance R/2 from the center of the arc. Find the magnitude and direction of magnetic field created by these two currents at point O. (see the figure below).



$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

$$d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{d\vec{s} \times \vec{r}}{r^3}$$

$$B_2 \, 2\pi r = \mu_0 i_2$$

$$\vec{B}_2 = \frac{\mu_0 i_2}{2\pi r} \otimes \; , \; B_2\left(r = \frac{R}{2}\right) = \frac{\mu_0 i_2}{\cancel{2}\pi R/\cancel{2}} \otimes$$

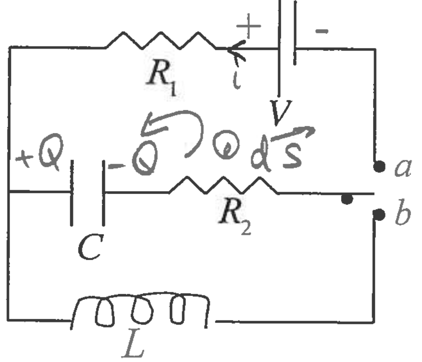
$$d\vec{B}_1 = \frac{\mu_0 i_1}{4\pi} \frac{ds R}{R^3} \odot$$

$$\vec{B}_1 = \frac{\mu_0 i_1}{4\pi R^2} R\theta \odot = \frac{\mu_0 i_1}{4\pi R} \theta \odot$$

$$\vec{B}_{total} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 i_1}{4\pi R} \theta \odot + \frac{\mu_0 i_2}{\pi R} \otimes$$

Problem 6: (15 points)

a) In a circuit below, R_1, R_2, C, L, and V are given. The switch is kept at a for a long time. Find the current through the resistor and the charge on the capacitor. *The problem will not be graded without a direction of current and charges on the capacitor indicated on the circuit.*



$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

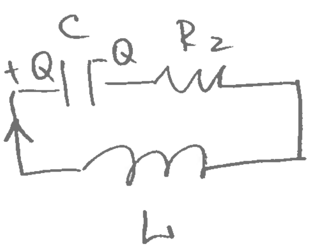
$$-V + \overset{=0}{i}(R_1+R_2) + \frac{Q}{C} = 0$$

$$\boxed{\begin{aligned} i &= 0 \\ Q &= CV \end{aligned}}$$

b) At $t = 0$ the switch is thrown to position b. Starting from some famous law, derive the equation that describes charge Q on the capacitor as a function of time. *The problem will not be graded without a direction of current and charges on the capacitor indicated on the circuit.*

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\left[\int \vec{B}\cdot d\vec{S}\right] = \Phi$$

$$i = \frac{dQ}{dt}$$



$$\Phi = \pm Li = -Li$$

$$\frac{d\Phi}{dt} = -L\frac{di}{dt}$$

$$\boxed{L\frac{d^2Q}{dt^2} + R_2\frac{dQ}{dt} + \frac{1}{C}Q = 0}$$

$$-\frac{Q}{C} - iR_2 = L\frac{di}{dt}$$

c) Neglect R_2. Solve for the charge on the capacitor as a function of time.

$$L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0$$

$$\boxed{\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0}$$

$$Q(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$Q(t=0) = A = CV$$

$$i = \frac{dQ}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$i(t=0) = B\omega = 0 \Rightarrow B = 0$$

$$Q(t) = CV\cos(\omega t)$$

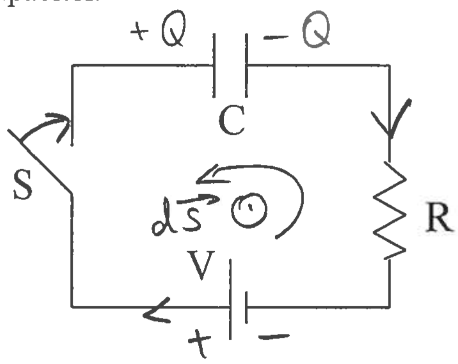
$$-A\omega^2\cos(\omega t) + \frac{A}{LC}\cos(\omega t) = 0$$

$$\omega^2 = \frac{1}{LC}$$

$$Q(t) = CV\cos\left(\frac{1}{\sqrt{LC}}t\right)$$

Problem 7: (15 points)

In the circuit below V, C, and R are given. At time $t = 0$, switch S is closed to begin charging the capacitor.



$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

a) Find the charge on the capacitor as a function of time and the current in the circuit. Neglect self-inductance. *The problem will not be graded without a direction of current and charges on the capacitor indicated on the circuit.*

$\Phi = 0$

$V - iR - \frac{Q}{C} = 0 \qquad i = \frac{dQ}{dt}$

$V - R\frac{dQ}{dt} - \frac{1}{C}Q = 0$

$$\boxed{R\frac{dQ}{dt} + \frac{1}{C}Q = V}$$

$Q(t) = Q_{ss} + Q_h$

$Q_{ss} = CV$

$Q_h = \alpha e^{-\beta t}$

$R\alpha(-\beta)e^{-\beta t} + \frac{1}{C}\alpha e^{-\beta t} = 0$

$$\boxed{\beta = \frac{1}{RC}}$$

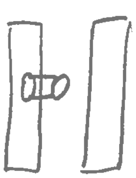
$Q(t) = CV + \alpha e^{-\frac{t}{RC}}$

$Q(t=0) = CV + \alpha = 0$

$\alpha = -CV$

$$\boxed{\begin{aligned} Q(t) &= CV\left(1 - e^{-\frac{t}{RC}}\right) \\ i(t) &= \frac{dQ}{dt} = \frac{V}{R}e^{-\frac{t}{RC}} \end{aligned}}$$

b) Find the electric field between the plates of the capacitor as a function of time. The area of the plates of the capacitor is A.



$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$

$E a = \frac{\sigma a}{\varepsilon_0}$

$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} = \frac{CV\left(1 - e^{-\frac{t}{RC}}\right)}{A\varepsilon_0}$

c) Find the displacement current between the plates.

$i_D = \varepsilon_0 \frac{d\Phi_E}{dt} = -\frac{\cancel{\varepsilon_0} CV}{\cancel{\varepsilon_0}}\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}} = \boxed{\frac{V}{R}e^{-\frac{t}{RC}}}$

$\Phi_E = \int \vec{E}\cdot d\vec{S} = \frac{CV\left(1 - e^{-\frac{t}{RC}}\right)}{\cancel{A}\varepsilon_0}\cancel{A}$

$$i = \frac{dQ}{dt}$$

$$R = \frac{V}{i}$$

$$R = \rho \frac{l}{A}$$

$$\vec{E} = \rho \vec{j}$$

$$i = \int_S \vec{j} \cdot d\vec{S}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i(d\vec{s} \times \vec{r})}{r^3}$$

$$d\vec{F} = i d\vec{s} \times \vec{B}$$

$$\Phi_B = \int \vec{B} \cdot d\vec{S}$$

$$\Phi_B = \pm Li$$

$$C = \frac{Q}{\Delta V}$$

$$\left|\vec{F}_k\right| = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{x^2}$$

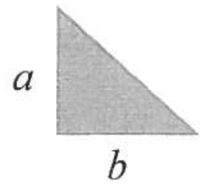
$$V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$

$$i_D = \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{S}$$

$$Q = \int \rho \, dV$$

$$Q = \int \sigma \, dS$$

$$Q = \int \lambda \, dx$$



$$S = \frac{1}{2} ab$$

$$\cos 45° = \frac{\sqrt{2}}{2}$$

$$\sin 45° = \frac{\sqrt{2}}{2}$$

$$\tan 45^{\circ} = 1$$