2007

Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

$$\oint \vec{B} \cdot d\vec{s} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{s}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{s}$$

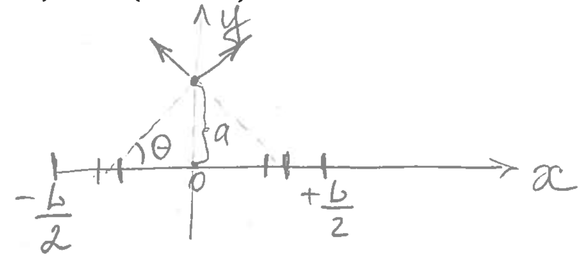
Problem 2: (15 points)

A rod of length L has a total charge Q smeared uniformly over it. A test charge q is a distance a away from the rod's midpoint.

a) What is the force that the rod exerts on the test charge?

$$\int \frac{dx}{(x^2+c)^{\frac{3}{2}}} = \frac{x}{c(x^2+c)^{\frac{1}{2}}} + const \qquad \int \frac{xdx}{(x^2+c)^{\frac{3}{2}}} = \frac{-1}{(x^2+c)^{\frac{1}{2}}} + const$$

$$d\vec{F} = \frac{q\,dQ}{4\pi\varepsilon_0 r^2}\hat{r}$$



Charge per unit length: $\frac{Q}{L}$

Charge of element dx:

$$dQ = \frac{Q}{L}dx$$

$$F_x = 0$$

$$F_y = 2\int_0^{L/2} \frac{Q}{L}\frac{q\,dx}{4\pi\varepsilon_0(x^2+a^2)}\sin\theta$$

$$r^2 = x^2+a^2; \quad \sin\theta = \frac{a}{r} = \frac{a}{\sqrt{x^2+a^2}}$$

$$F_y = 2\int_0^{L/2} \frac{Qqa}{4\pi\varepsilon_0 L}\frac{dx}{(x^2+a^2)^{3/2}} = \frac{Qqa}{2\pi\varepsilon_0 L}\frac{x}{a^2(x^2+a^2)^{\frac{1}{2}}}\Bigg|_0^{L/2} =$$

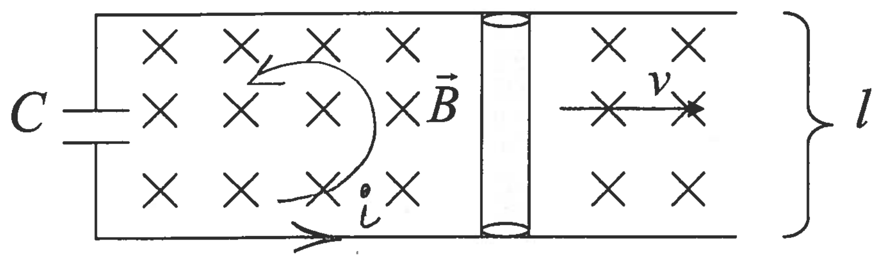
$$= \frac{Qq}{4\pi\varepsilon_0}\frac{1}{a\left(\frac{L^2}{4}+a^2\right)^{\frac{1}{2}}}; \quad \boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{a\sqrt{\frac{L^2}{4}+a^2}}\vec{i}_y}$$

b) What is the force at $a>>L$?

$$a \gg L \qquad \boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{a^2}\vec{i}_y}$$

Problem 3: (20 points)

a) A rod with resistance R is pushed with a velocity v along the two parallel, horizontal rails that are a distance l apart. The rails have no resistance and are connected by a capacitor C. There is a constant uniform magnetic field as shown. If the capacitor has zero charge at $t=0$, find the charge on the capacitor as a function of time (<u>determine the equation for the charge, do not solve it</u>). Ignore the self-inductance of the circuit.



$$\oint \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{s}$$

$$iR + \frac{1}{C}Q = -\frac{d}{dt}\left(-\int B\cdot ds\right) = \frac{d}{dt}(Blx) = Blv$$

$$i = \frac{dQ}{dt}$$

$$\boxed{R\frac{dQ}{dt} + \frac{1}{C}Q = Blv}$$

b) In part a) suppose that the self-inductance is L. Write the equation for the charge on the capacitor's plates.

$$\Phi_L = +Li \; ; \; \frac{d\Phi_L}{dt} = +L\frac{di}{dt} = +L\frac{d^2Q}{dt^2}$$

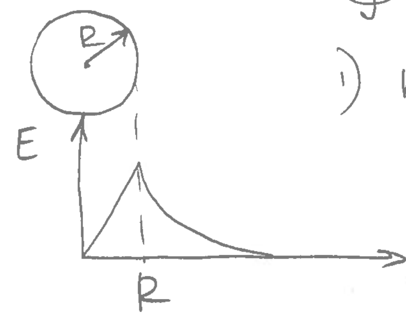
$$iR + \frac{1}{C}Q = Blv - \frac{d\Phi_L}{dt} = Blv - L\frac{d^2Q}{dt^2}$$

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = Blv}$$ (we neglect $\frac{dL}{dt}$)

Problem 4: (20 points)

a) An insulating sphere of radius R has charge **uniformly** spread through it with a charge density ρ . Find the electric field everywhere.

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$



1) $r<R$ $\quad E\,4\pi r^2 = \frac{1}{\varepsilon_0}\int \rho\, dv = \frac{1}{\varepsilon_0}\rho \frac{4}{3}\pi r^3$

$$\boxed{E = \frac{\rho}{3\varepsilon_0} r} \quad r<R$$

2) $r>R$ $\quad E\,4\pi r^2 = \frac{\int \rho\, dv}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}\frac{4}{3}\pi R$

$$\boxed{E = \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}} \quad r>R$$

b) Find the difference in an electric potential between the center and a point $4R$.

$$V(0) - V(4R) = -\int_{4R}^{0} \vec{E}\cdot d\vec{r} = \int_0^R \frac{\rho r}{3\varepsilon_0} dr + \int_R^{4R} \frac{\rho R^3}{3\varepsilon_0 r^2} dr$$

$$= \frac{\rho R^2}{6\varepsilon_0} - \frac{\rho R^3}{12\varepsilon_0 R} + \frac{\rho R^3}{3\varepsilon_0 R} = \frac{5}{12}\frac{\rho R^2}{\varepsilon_0}$$

c) If ρ is a function of r, $\rho = \alpha r$, where r is a distance from the center of the sphere, $\alpha = const$, find the electric field everywhere.

$$\oint \vec{E}\cdot d\vec{S} = \frac{\int \rho\, dv}{\varepsilon_0}$$

$r<R$ $\quad E\cdot 4\pi r^2 = \frac{\int_0^r \alpha r\, 4\pi r^2 dr}{\varepsilon_0} = \frac{4\pi\alpha}{\varepsilon_0}\frac{r^4}{4} = \frac{\pi\alpha r}{\varepsilon_0}$

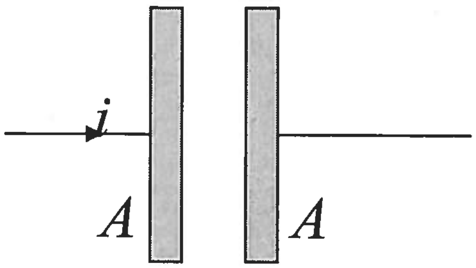
$$\boxed{E = \frac{\alpha r^2}{4\varepsilon_0}} \quad r<R$$

$r>R$ $\quad E\cdot 4\pi r^2 = \frac{\int_0^R \alpha r\, 4\pi r^2 dr}{\varepsilon_0} = \frac{4\pi\alpha R^4}{4\varepsilon_0} = \frac{\pi\alpha R}{\varepsilon_0}$

$$\boxed{E = \frac{\alpha R^4}{4\varepsilon_0 r^2}} \quad r>R$$

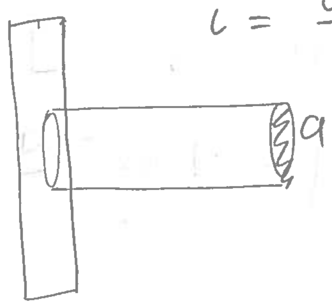
Problem 5: (15 points)

Consider two parallel plates of area A in some circuit:



The electric field is a function of time $E = E_0 \cos \omega t$. Find the current i in the wire and show that it is equal to the displacement current between the plates.

$$i = \frac{dQ}{dt}$$



$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

$$E \cdot \cancel{a} = \frac{Q}{A\varepsilon_0} \cancel{a}$$

$$Q = EA\varepsilon_0 = A\varepsilon_0 E_0 \cos \omega t$$

$$i = \frac{dQ}{dt} = -\omega A \varepsilon_0 E_0 \sin \omega t$$

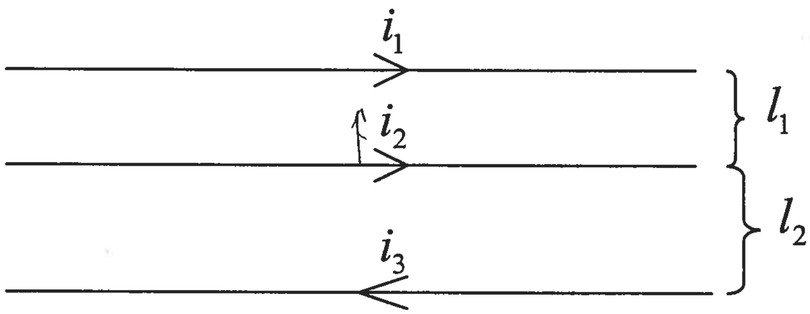
$$i_D = \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\Phi_E = \int \vec{E} \cdot d\vec{S} = E_0 \cos \omega t \cdot A$$

$$i_D = \varepsilon_0 \frac{d}{dt} (A E_0 \cos \omega t) = -\omega \varepsilon_0 A E_0 \sin \omega t \,!$$

Problem 6: (15 points)

Consider three infinitely long wires in the plane of the paper (i_3 is in the opposite direction with respect to i_1 and i_2).



Find the force exerted on a length l of the center wire.

$$d\vec{F} = i\, d\vec{s} \times \vec{B}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_3$$

$$\vec{B}_1 = ?$$

$$\oint \vec{B}_1 \cdot d\vec{r} = \mu_0 i_1$$

$$B_1 2\pi r = \mu_0 i_1$$

$$B_1 = \frac{\mu_0 i_1}{2\pi r}$$

$$\vec{B}_1 = \frac{\mu_0 i_1}{2\pi l_1} \otimes$$

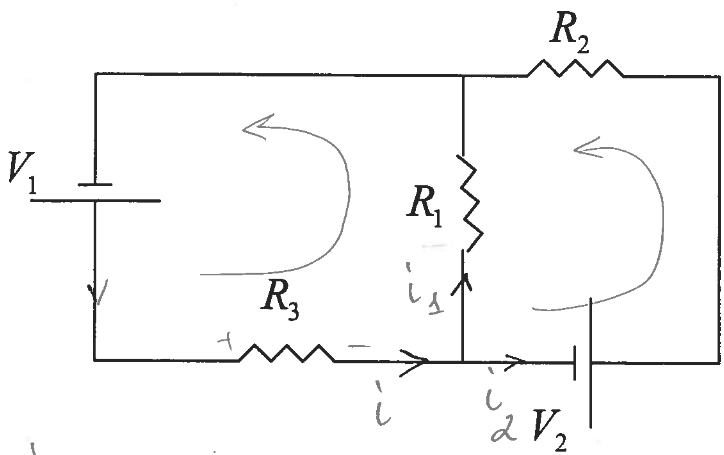
$$\vec{B}_3 = \frac{\mu_0 i_3}{2\pi l_2} \otimes$$

$$\vec{B} = \frac{\mu_0}{2\pi}\left(\frac{i_1}{l_1} + \frac{i_3}{l_2}\right) \otimes$$

$$\vec{F} = i_2\, l\, \frac{\mu_0}{2\pi}\left(\frac{i_1}{l_1} + \frac{i_3}{l_2}\right) \text{ up}$$

Problem ~~8~~ 7: (15 points)

a) Write Kirchhoff's rules for this circuit below.



$$\begin{cases} i = i_1 + i_2 \\ V_1 - iR_3 - i_1 R_1 = 0 \\ V_2 - i_2 R_2 + i_1 R_1 = 0 \end{cases}$$

b) Find the current through resistor R_2 assuming that $R_1 = R_2 = R_3 = R$ and V_1 and V_2 are given.

$$i_2 = \frac{V_2}{R} + i_1$$

$$V_1 - i_1 R - i_2 R - i_1 R = 0$$

$$i_2 = \frac{V_1 - 2 i_1 R}{R} = \frac{V_1}{R} - 2 i_1 = \frac{V_1}{R} - 2 i_2 + 2\frac{V_2}{R}$$

$$3 i_2 = \frac{V_1 + 2V_2}{R}$$

$$\boxed{i_2 = \frac{V_1 + 2V_2}{3R}}$$

c) Find the voltage drop across the resistor R_3 assuming that $R_1 = R_2 = R_3 = R$ and V_1 and V_2 are given.

$$i = i_1 + i_2 = 2i_2 - \frac{V_2}{R} = \frac{2}{3}\frac{V_1 + 2V_2}{R} - \frac{V_2}{R} = \frac{2V_1 + V_2}{3R}$$

$$\boxed{V = R \cdot i = \frac{2V_1 + V_2}{3}}$$