Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

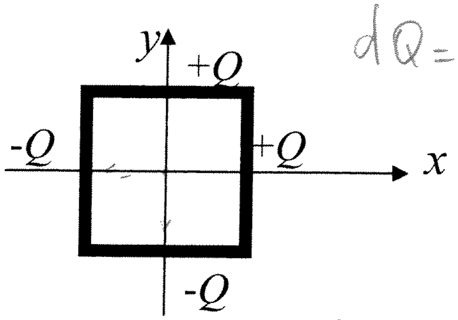
$$\oint \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{S}$$

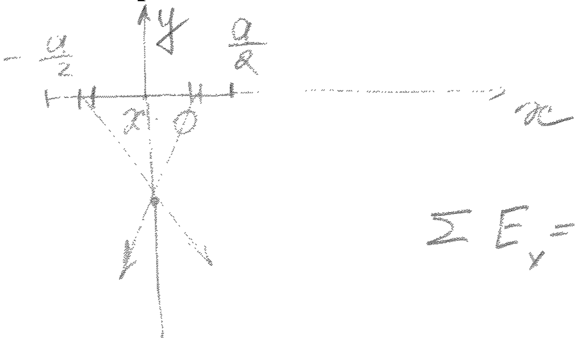
Problem 2: (15 points)

Electric charge is distributed uniformly along each side of a square with length a. Two adjacent sides have positive charge with total charge $+Q$ on each.

a) If the other two sides have negative charge with total charge $-Q$ on each (see the figure below), what is the net electric field at the center of the square?



$$dQ = \frac{Q}{a}\,dx$$



$$\sum E_x = 0$$

$$dE_y = -\frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2}\sin\theta$$

$$\sin\theta = \frac{a}{2r}\,; \quad r = \sqrt{x^2 + \frac{a^2}{4}}$$

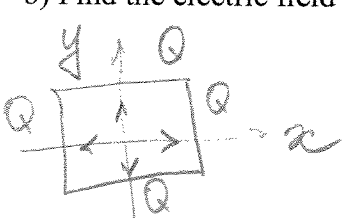
$$E_y = -\frac{2}{4\pi\varepsilon_0}\int_0^{\frac{a}{2}} \frac{Q}{a}\,\frac{a\,dx}{2\left(x^2+\frac{a^2}{4}\right)^{3/2}} =$$

$$= -\frac{Q}{4\pi\varepsilon_0}\,\frac{x}{\frac{a^2}{4}\left(x^2+\frac{a^2}{4}\right)^{1/2}}\Bigg|_0^{\frac{a}{2}} = -\frac{Q\,\frac{a}{2}}{\pi\varepsilon_0 a^2\left(\frac{2a^2}{4}\right)^{1/2}} =$$

$$= -\frac{Q}{\sqrt{2}\,\varepsilon_0\pi a^2}$$

$$\vec{E} = -\frac{\sqrt{2}}{\varepsilon_0\pi}\,\frac{Q}{a^2}\left(\vec{i}_x + \vec{i}_y\right)$$

b) Find the electric field in the center of the square if all four sides have positive charge $+Q$.



$$\vec{E} = 0$$

Problem 3: (20 points)

a) A non-uniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right) \quad for\ r \le R$$

$$\rho(r) = 0 \quad for\ r \ge R$$

Find the electric field everywhere.

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$Q_{encl} = \int \rho\, dV;\quad dV = 4\pi r^2 dr$$

$r<R$ $\quad Q_{encl} = \int_0^r \rho_0\left(1-\frac{r}{R}\right)4\pi r^2 dr = \rho_0\left(\int_0^r 4\pi r^2 dr - \int_0^r \frac{r}{R}4\pi r^2 dr\right.$

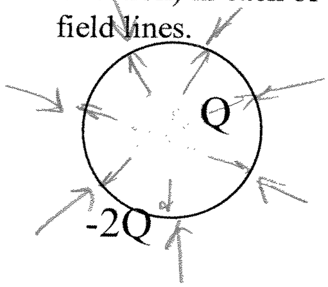
$= \rho_0\left(4\pi\frac{r^3}{3} - \frac{4\pi r^4}{4R}\right);\quad E\cdot 4\pi r^2 = \frac{\rho_0}{\varepsilon_0}\left(\frac{4\pi r^3}{3} - \frac{4\pi r^4}{4R}\right)$

$E = \frac{\rho_0}{4\pi\varepsilon_0 r^2}\left(\frac{4\pi r^3}{3} - \frac{4\pi r^4}{4R}\right) = \boxed{\frac{\rho_0}{\varepsilon_0 r^2}\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)}$ radially out

$r>R$ $\quad Q_{encl} = \int_0^R \rho_0\left(1-\frac{r}{R}\right)4\pi r^2 dr = \rho_0\left(\frac{4\pi R^3}{3} - \frac{4\pi R^4}{4R}\right) = \frac{4\pi\rho_0 R^3}{12}$

$Q_{encl} = \frac{\pi\rho_0 R^3}{3};\quad E\,4\pi r^2 = \frac{\pi\rho_0 R^3}{3\varepsilon_0};\quad \boxed{E = \frac{\rho_0 R^3}{12\varepsilon_0 r^2}}$ radially out

b) A solid conducting sphere with radius R, that carries positive charge Q, is concentric with a very thin conducting shell of radius 2R that carries charge -2Q. Find the electric field (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$. Draw schematically the electric field lines.



$r<R \qquad E=0$

$R<r<2R \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{i}_r$

$r>2R \qquad \vec{E} = -\frac{Q}{4\pi\varepsilon_0 r^2}\vec{i}_r$

c) For part b) find a difference in electric potential between points $r = 0$ and $r = \infty$

$V(0) - V(\infty) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r} = \int_{\infty}^{2R} \frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_{2R}^{R} \frac{Q}{4\pi\varepsilon_0 r^2}dr =$

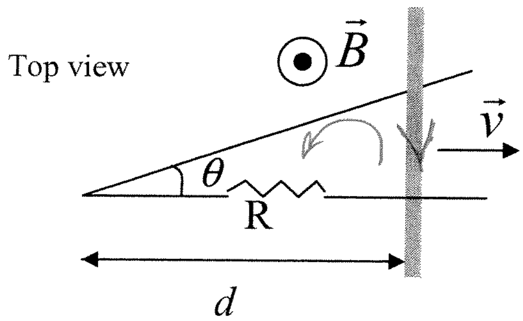
$= -\frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\Big|_{\infty}^{2R} + \frac{Q}{4\pi\varepsilon_0 r}\Big|_{2R}^{R} = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{2R} + \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 2R} =$

$= 0$

Problem 4: (20 points)

Two horizontal conducting rails form an angle θ where their ends are joined. A conducting rod slides on the rails without friction with constant velocity v as shown in the figure. At t=0 it starts from x=d from rest. There is a resistor R in one of the rails. There is a uniform magnetic field in the direction shown.

(a) Find the current as a function of time. Ignore self-inductance.



$$S = \frac{1}{2} x x \tan\theta = \frac{1}{2} x^2 \tan\theta$$

$$\Phi = \int \vec{B} \cdot d\vec{S} = B \frac{1}{2} x^2 \tan\theta$$

$$x = d + vt$$

$$\Phi = B \frac{1}{2}(d+vt)^2 \tan\theta$$

$$\frac{d\Phi}{dt} = B(d+vt)v\tan\theta$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi}{dt}$$

$$-iR = -B(d+vt)v\tan\theta$$

$$i = \frac{B(d+vt)v\tan\theta}{R} \quad \text{clockwise}$$

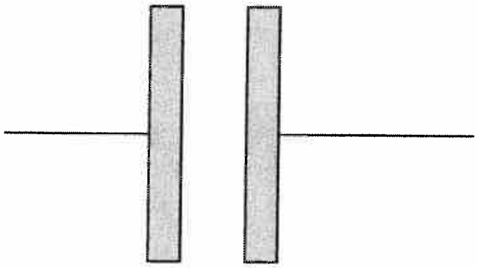
b) Find the force that needs to be applied to move the rod with constant velocity.

$$\vec{F} = i\vec{\ell} \times \vec{B} = i\ell B = (d+vt)\tan\theta\, iB\, \vec{i}_x$$

$$\ell = x \cdot \tan\theta = (d+vt)\tan\theta$$

Problem 5: (15 points)

Show that the displacement current between the plates of a parallel plate capacitor is given by $C\frac{dV}{dt}$, where C is capacitance and V is the potential difference between the plates.



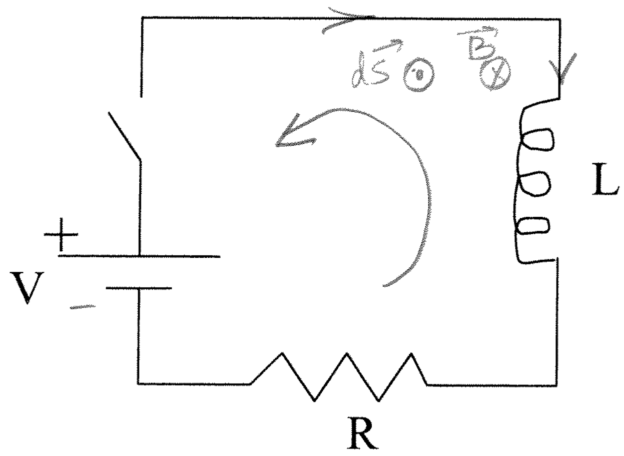
$$i_D = \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{S} = \varepsilon_0 \frac{d}{dt} E \cdot A =$$

$$= \varepsilon_0 A \frac{dE}{dt} = \varepsilon_0 A \frac{d}{dt} \frac{V}{d} =$$

$$= \frac{\varepsilon_0 A}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

Problem 6: (15 points)

In the circuit below the switch is closed at t=0.



Find the current as a function of time and plot it schematically

$$\oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi}{dt}$$

$$\Phi = \pm L i; \quad \Phi = -Li; \quad \frac{d\Phi}{dt} = -L\frac{di}{dt}$$

$$V - iR = +L\frac{di}{dt}$$

$$L\frac{di}{dt} + Ri = V$$

$$i(t) = \frac{V}{R} + \alpha e^{-\frac{R}{L}t}$$

$$L\frac{di}{dt} + Ri = 0$$

$$i = \alpha e^{-\beta t}$$

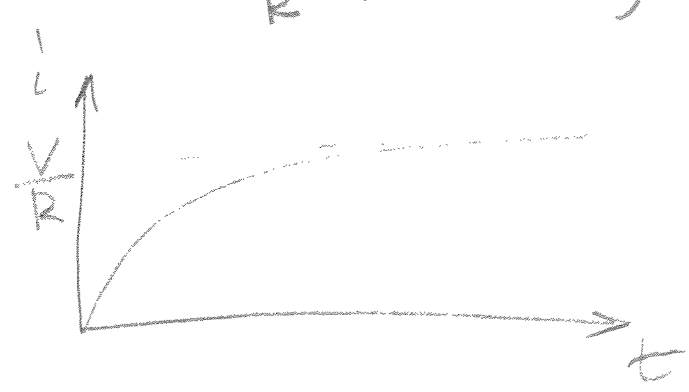
$$-\alpha L \beta e^{-\beta t} + R\alpha e^{-\beta t} = 0$$

$$\beta = \frac{R}{L}$$

$$i(0) = 0 = \frac{V}{R} + \alpha = 0$$

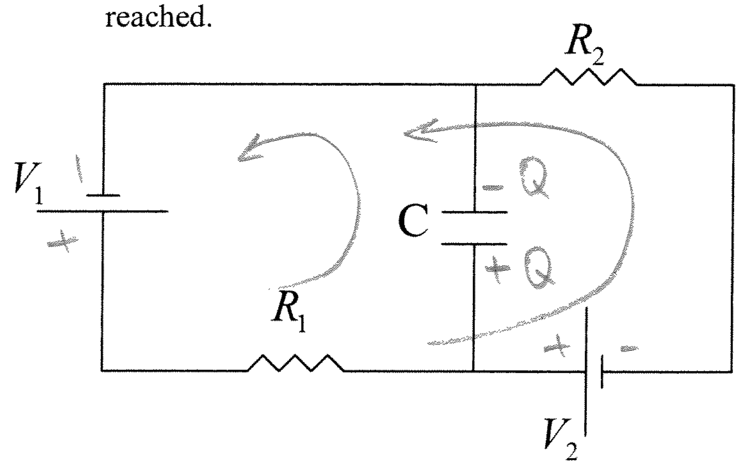
$$\alpha = -\frac{V}{R}$$

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$



Problem 7: (15 points)

The circuit below was put together a long time ago so that the steady state has been reached.



Find all currents and the charge on the capacitor.

$$\begin{cases} -V_1 + iR_1 + V_2 + iR_2 = 0 \\ -V_1 + iR_1 + \frac{Q}{C} = 0 \end{cases} \Rightarrow i = \frac{V_1 - V_2}{R_1 + R_2}$$

$$\frac{Q}{C} = V_1 - iR_1 = V_1 - \frac{V_1 - V_2}{R_1 + R_2} R_1$$