Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

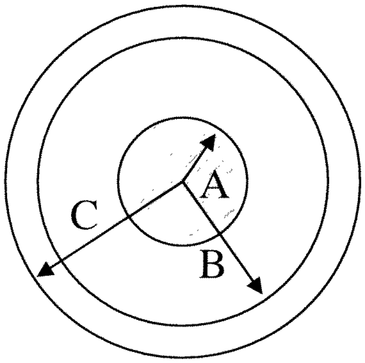
$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

Problem 2: (20 points)

A solid sphere of radius A has a net charge Q uniformly spread throughout. It is surrounded by a conducting spherical shell with inner radius B and outer radius C.

a) Find the difference in the electric potential between a point at the center of the sphere and infinity.



$$V(0)-V(\infty) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r}$$

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$r<A$ $\quad E\,4\pi r^2 = \frac{1}{\varepsilon_0}\frac{Q}{\frac{4}{3}\pi A^3}\frac{4}{3}\pi r^3$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{A^3}r \quad \text{radially out}$$

$A<r<B$ $\quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$; $\quad B<r<C$ $\quad E=0$

$r>C$ $\quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$

$$V(0)-V(\infty) = -\left[\int_{\infty}^{C}\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr + \int_{C}^{B}0\,dr + \int_{B}^{A}\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr + \int_{A}^{0}\frac{1}{4\pi\varepsilon_0}\frac{Q}{A^3}r\,dr\right] = -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{C}-\frac{1}{A}+\frac{1}{B}-\frac{1}{2A}\right]$$

b) If, instead, the sphere of radius A has a non-uniform, but spherically symmetric, distribution of charge with charge density $\rho(r)=\rho_0\frac{r^2}{A^2}$, find the electric field everywhere (r<A, A<r<B, B<r<C, r>C).

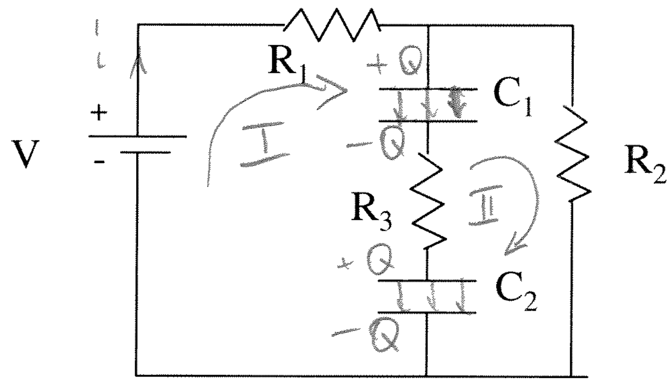
$r<A$ $\quad E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^r \rho_0\frac{r^2}{A^2}4\pi r^2 dr = \frac{4\pi\rho_0}{\varepsilon_0 A^2}\frac{r^5}{5}$

$A<r<B$ $\quad E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^A \rho_0\frac{r^2}{A^2}4\pi r^2 dr = \frac{4\pi\rho_0}{\varepsilon_0 A^2}\frac{A^5}{5}$

$$E = \begin{cases} \frac{\rho_0}{5\varepsilon_0 A^2}r^3, & r<A \\ \frac{\rho_0 A^3}{5\varepsilon_0}\frac{1}{r^2}, & A<r<B \\ 0, & B<r<C \\ \frac{\rho_0 A^3}{5\varepsilon_0}\frac{1}{r^2}, & r>C \end{cases} \quad \text{radially out}$$

Problem 3: (15 points)

The circuit below was put together a long time ago so that the steady state has been reached.



Find all currents in the circuit and the charges on the capacitors.

$$\oint \vec{E} \cdot d\vec{r} = 0$$

$$-V + iR_1 + iR_2 = 0$$

$$i = \frac{V}{R_1 + R_2}$$

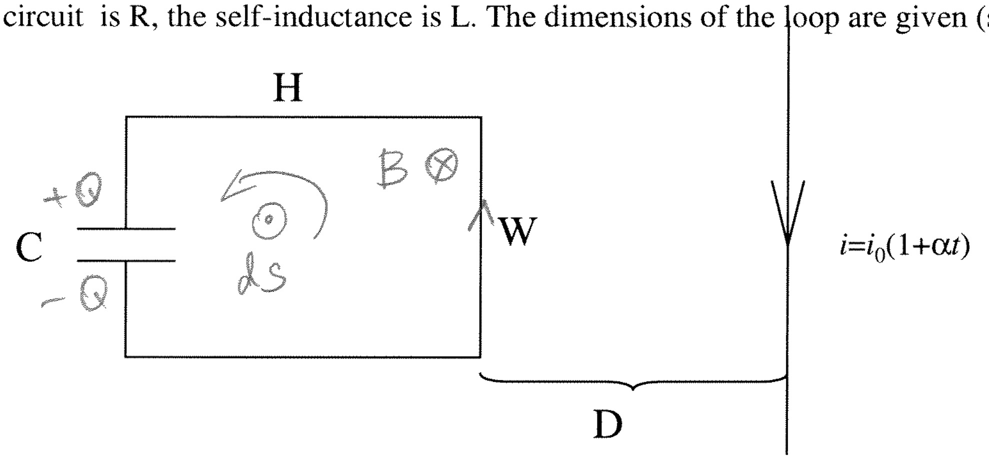
II : $$iR_2 - \frac{Q}{C_2} - \frac{Q}{C_1} = 0$$

$$iR_2 - Q\left(\frac{1}{C_2} + \frac{1}{C_1}\right) = 0$$

$$Q = iR_2 \frac{C_2 C_1}{C_1 + C_2} = V \frac{R_2}{R_1 + R_2} \frac{C_2 C_1}{C_1 + C_2}$$

Problem 4: (20 points)

There is an infinitely long wire with time-dependent current $i=i_0(1+\alpha t)$ where i_0 and α are known constants. It is placed near a circuit with a capacitor of capacitance C. The resistance of the wires in the circuit is R, the self-inductance is L. The dimensions of the loop are given (see the Figure).



a) What is the equation that would have to be solved to find the charge on the capacitor? DO NOT SOLVE IT

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 i \;; \quad B\,2\pi r = \mu_0 i \;; \quad \vec{B} = \frac{\mu_0 i_0(1+\alpha t)}{2\pi r} \otimes$$

$$\Phi = -\int_D^{D+H} \frac{\mu_0 i_0(1+\alpha t)}{2\pi r} W\,dr = -\frac{\mu_0 i_0(1+\alpha t)}{2\pi} W \ln\frac{D+H}{D}$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial \Phi}{\partial t} \;; \quad \frac{\partial \Phi}{\partial t} = -\frac{\mu_0 i_0 \alpha}{2\pi} W \ln\frac{D+H}{D}$$

$$\Phi_{self} = Li \;; \quad iR + \frac{Q}{C} = \frac{\mu_0 i_0 \alpha}{2\pi} W \ln\frac{D+H}{D} - L\frac{di}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \frac{\mu_0 i_0 \alpha W}{2\pi}\ln\frac{D+H}{D}$$

b) Find the charge on the capacitor as a function of time, ignoring the self inductance of the loop and assuming that at $t=0$ the capacitor was uncharged.

$$R\frac{dQ}{dt} + \frac{1}{C}Q = \frac{\mu_0 i_0 \alpha W}{2\pi}\ln\frac{D+H}{D} = \mathcal{E}$$

$Q(t) = (1) + (2)$

(1): $Q = C\mathcal{E}$

(2): $R\frac{dQ}{dt} + \frac{1}{C}Q = 0$

$Q(t) = \alpha e^{-\beta t}$

$-R\alpha\beta e^{-\beta t} + \frac{1}{C}\alpha e^{-\beta t} = 0$

$\beta = \frac{1}{RC}$

$Q(t) = C\mathcal{E} + \alpha e^{-\frac{t}{RC}}$

$Q(t=0) = C\mathcal{E} + \alpha = 0$

$\alpha = -C\mathcal{E}$

$Q(t) = C\mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)$

Problem 5: (15 points)

a) Which of the following equations is the wave equation?

$$\frac{\partial E_y}{\partial x} = \sqrt{\mu_0 \varepsilon_0}\,\frac{\partial E_y}{\partial t} \qquad \frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

b) Find the relationship between α and β for $E_y = A\sin(\alpha x - \beta t)$ to be a solution of the wave equation.

$$\frac{\partial E_y}{\partial x} = A\alpha \cos(\alpha x - \beta t)$$

$$\frac{\partial^2 E_y}{\partial x^2} = -A\alpha^2 \sin(\alpha x - \beta t)$$

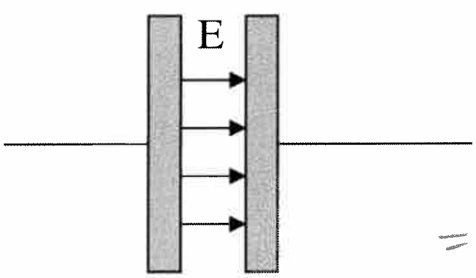
$$\frac{\partial E_y}{\partial t} = -A\beta \cos(\alpha x - \beta t)$$

$$\frac{\partial^2 E_y}{\partial t^2} = -A\beta^2 \sin(\alpha x - \beta t)$$

$$-A\alpha^2 = -\mu_0 \varepsilon_0 A \beta^2$$

$$\boxed{\mu_0 \varepsilon_0 = \frac{\alpha^2}{\beta^2}}$$

c) Find the displacement current if the electric field between parallel plate capacitor is $E=E_0\sin(\omega t+\gamma)$. The area of the plates is A.



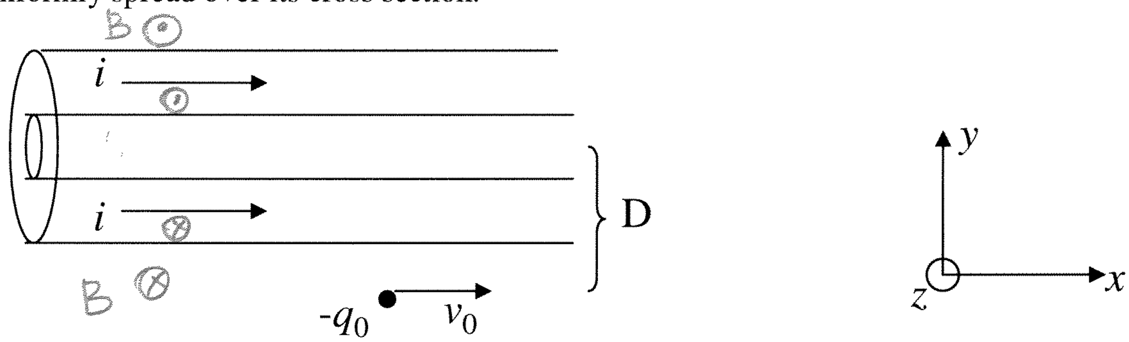
$$i_D = \varepsilon_0 \frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{S} =$$

$$= \varepsilon_0 \frac{\partial}{\partial t}\int E_0 \sin(\omega t + \gamma)\cdot dS =$$

$$= \varepsilon_0 E_0 \omega \cos(\omega t + \gamma)\cdot A$$

Problem 6: (15 points)

A very long, hollow cylindrical wire, inner radius A and outer radius B, carries a constant current i, uniformly spread over its cross section.



a) Find the magnetic field everywhere (r<A; A<r<B; r>B)

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

$r < A \quad B = 0$

$A < r < B \quad B\, 2\pi r = \mu_0 \frac{i}{\pi B^2 - \pi A^2}(\pi r^2 - \pi A^2)$

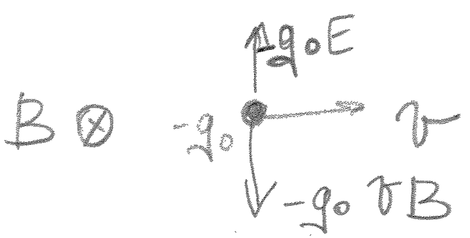
$$B = \frac{\mu_0 i}{2\pi r} \frac{r^2 - A^2}{B^2 - A^2}$$

$r > B \quad B\, 2\pi r = \mu_0 i$

$$B = \frac{\mu_0 i}{2\pi r}$$

b) A **negatively** charged particle, charge $-q_0$, is moving at the distance D from the center of the wire parallel to the axis of the wire with a constant velocity v_0. What constant electric field would have to be applied for the particle to experience no net force? Ignore gravity.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = 0$$



$q_0 E = q_0 v B$

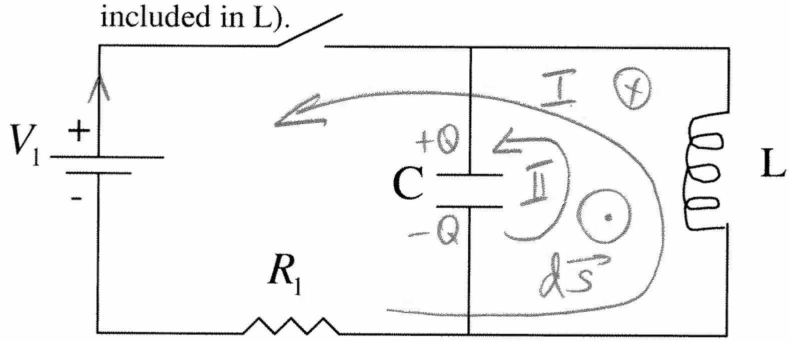
$E = v B$

$\vec{E} = - v B \vec{i_y}$

$$B = \frac{\mu_0 i}{2\pi D}$$

Problem 7: (15 points)

In the circuit below the switch has been **closed** for a long time. If at t=0 the switch is opened, find the charge on the capacitor plates as a function of time. (All self-inductance is included in L).



$$\oint \vec{E}\cdot d\vec{r} = 0$$

$$\mathrm{I}:\ V - iR_1 = 0 \Rightarrow \boxed{i = \frac{V_1}{R_1}}$$

$$\mathrm{II}:\ \boxed{Q = 0}$$

Switch is opened

$$Q(t=0) = 0;\quad i(t=0) = \frac{V_1}{R_1}$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial \Phi}{\partial t};\quad \Phi = -Li$$

$$\frac{Q}{C} = L\frac{di}{dt};\quad i = -\frac{dQ}{dt}$$

$$\boxed{L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0}$$

$$Q(t) = A\cos\omega t + B\sin\omega t$$

$$i(t) = -\frac{dQ}{dt} = A\omega\sin\omega t - B\omega\cos\omega t$$

$$Q(t=0) = A = 0$$

$$i(t=0) = -B\omega = \frac{V_1}{R_1}$$

$$L(-A\omega^2\cos\omega t - B\omega^2\sin\omega t) + \frac{1}{C}(A\cos\omega t + B\sin\omega t) = 0$$

$$\boxed{\omega = \sqrt{\frac{1}{LC}} \qquad Q(t) = -\frac{V_1}{R_1\omega}\sin\omega t}$$