Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

Problem 2: (18 points)

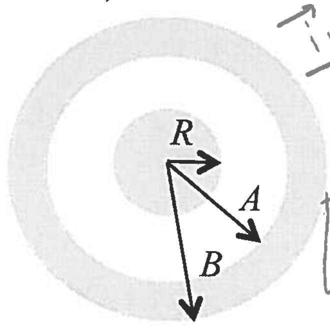
A non-uniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$\rho(r) = \rho_0(1 - r/R) \quad for \quad r \le R$

$\rho(r) = 0 \quad for \quad r \ge R$

where ρ_0 and R are known positive constants. It is surrounded by a conducting spherical shell, inner radius A and outer radius B. The spherical shell is charged with charge $-Q$.

a) Find the electric field at $r < R$, $R < r < A$, $A < r < B$, $r > B$



$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{encl}}{\varepsilon_0}$$

$\boxed{r<R}$ $$E\, 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^r \rho\, dV = \frac{1}{\varepsilon_0}\int_0^r \rho_0\left(1-\frac{r}{R}\right)4\pi r^2 dr$$

$$= \frac{1}{\varepsilon_0} 4\pi \rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right); \quad \boxed{\vec{E} = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)\vec{i}_r}$$

$\boxed{R<r<A}$ $$E\, 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^R \rho_0\left(1-\frac{r}{R}\right)dr = \frac{1}{\varepsilon_0} 4\pi \rho_0 \left(\frac{R^3}{3} - \frac{R^3}{4}\right) =$$

$$= \frac{4\pi \rho_0 R^3}{12\varepsilon_0}; \quad \boxed{\vec{E} = \frac{\rho_0 R^3}{12\varepsilon_0 r^2}\vec{i}_r}; \quad \boxed{A<r<B \quad E=0}$$

$r > B$ $$E\, 4\pi r^2 = \frac{\pi \rho_0 R^3}{3\varepsilon_0} - \frac{Q}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi r^2 \varepsilon_0}\left(\frac{\pi \rho_0 R^3}{3} - Q\right)\vec{i}_r$$

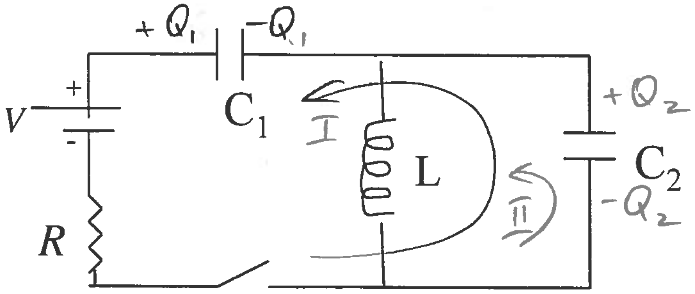
b) Find a difference in electric potential between $r = R$ and $r = B$: $V(B)-V(R)$.

$$V(B) - V(R) = -\int_R^B \vec{E}\cdot d\vec{r} = -\left[\int_R^A \frac{\rho_0 R^3}{12\varepsilon_0 r^2}dr + \int_A^B 0\right]$$

$$= \frac{\rho_0 R^3}{12\varepsilon_0 r}\Bigg|_R^A = \boxed{\frac{\rho_0 R^3}{12\varepsilon_0}\left(\frac{1}{A} - \frac{1}{R}\right)}$$

Problem 3: (18 points)

a) In the circuit below the switch has been **closed** for a long time so that the steady state has been reached. Find the current through the resistor and the charges on the capacitors.



$$\oint \vec{E}\cdot d\vec{r} = 0$$

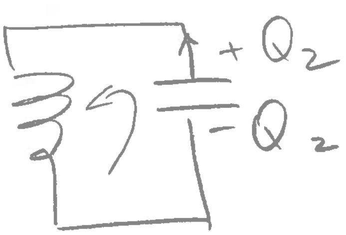
$$\boxed{i = 0}$$

$$\text{I}: \; V - \frac{Q_2}{C_2} - \frac{Q_1}{C_1} = 0$$

$$\text{II}: \; -\frac{Q_2}{C_2} = 0 \;\Rightarrow\; \boxed{Q_2 = 0}$$

$$\boxed{Q_1 = C_1 V}$$

b) At t=0 the switch is opened. Find Q_2, the charge on the capacitor C_2, as a function of time. (Assume that L includes self-inductance).



$$i(t=0) = 0 \quad Q_2(t=0) = 0$$

$$\boxed{Q_2(t) = 0}$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s} \qquad \Phi = Li$$

$$-\frac{Q_2}{C_2} = -L\frac{di}{dt} = L\frac{d^2 Q_2}{dt^2}; \qquad i = -\frac{dQ_2}{dt}$$

$$L\frac{d^2 Q_2}{dt^2} + \frac{1}{C_2}Q_2 = 0; \qquad \frac{d^2 Q}{dt^2} + \frac{1}{LC_2}Q_2 = 0$$

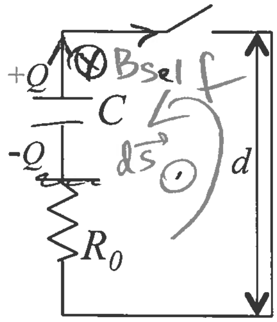
$$Q_2(t) = A\cos\omega t + B\sin\omega t$$

$$\omega = \frac{1}{\sqrt{LC_2}}; \qquad A = 0; \quad B = 0; \quad Q_2(t) = 0$$

Problem 4: (18 points)

In the circuit below, the capacitor has capacitance C and is initially charged to Q_0 with the polarity shown. R_0 is a known resistance. At time t=0 the switch is closed.

a) Find the charge Q and the current i as a function of time. Ignore self-inductance.



$i = -\frac{dQ}{dt}$; $\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s} = 0$

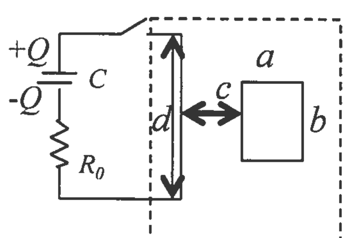
$\frac{Q}{C} - iR_0 = 0$; $\frac{Q}{C} + R_0\frac{dQ}{dt} = 0$

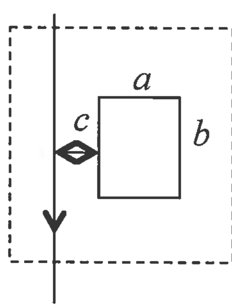
$\frac{dQ}{dt} + \frac{1}{R_0C}Q = 0$; $Q(t) = \alpha e^{-\beta t}$

$-\alpha\beta e^{-\beta t} + \frac{1}{R_0C}\alpha e^{-\beta t} = 0$; $\beta = \frac{1}{R_0C}$; $Q(t) = \alpha e^{-\frac{t}{R_0C}}$

$Q(t=0) = Q_0 = \alpha$; $\boxed{Q(t) = Q_0 e^{-\frac{t}{R_0C}}}$ $\boxed{i = -\frac{dQ}{dt} = \frac{Q_0}{R_0C}e^{-\frac{t}{R_0C}}}$

b) Now suppose that there is a small circuit, dimensions a by b (b<<d), which has a resistance R and is not connected in any way to the large circuit. The distance c is given. Assume that only the wire nearest the small circuit produces an appreciable magnetic field through it. The wire can be considered as infinitely long (the dashed part is shown on the right). Find the current i_1 (magnitude and direction) in the small circuit as a function of time using the current in the large circuit, found in part a). Ignore self-inductance.





$i(t) = \frac{Q_0}{R_0C}e^{-\frac{t}{R_0C}}$

$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\underbrace{\boxed{\int \vec{B}\cdot d\vec{S}}}_{\Phi}$

$\oint \vec{B}\cdot d\vec{r} = \mu_0 i$; $B 2\pi r = \mu_0 i$

$B = \frac{\mu_0 i}{2\pi r}$

$\Phi = \int_c^{c+a}\frac{\mu_0 i}{2\pi r} b\,dr = \frac{\mu_0 i b}{2\pi}\ln\frac{c+a}{c}$

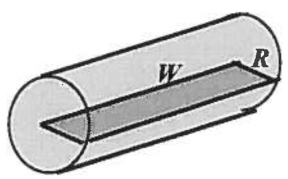
$\frac{d\Phi}{dt} = \frac{\mu_0 b}{2\pi}\ln\frac{c+a}{c}\,\frac{d}{dt}\left(\frac{Q_0}{R_0C}e^{-\frac{t}{R_0C}}\right) =$

$= -\frac{\mu_0 b}{2\pi}\ln\frac{c+a}{c}\,\frac{Q_0}{(R_0C)^2}e^{-\frac{t}{R_0C}}$

$i_1 R = \frac{\mu_0 b}{2\pi}\frac{Q_0}{(R_0C)^2}\ln\frac{c+a}{c}e^{-\frac{t}{R_0C}}$; $\boxed{i_1(t) = \frac{\mu_0 b}{2\pi R}\frac{Q}{(R_0C)^2}\ln\frac{c+a}{c}e^{-\frac{t}{R_0C}}}$

Problem 5: (18 points)

A very long, cylindrical wire of radius R carries a current i_0 uniformly distributed across the cross section of the wire and pointing into the page.



$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

a) Derive an expression for the magnetic field at $r < R$, and $r > R$.

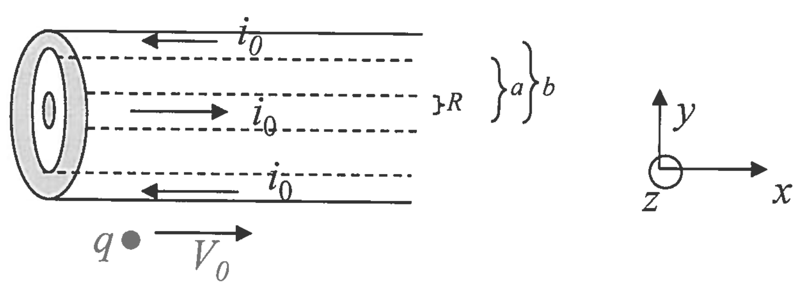
$r < R$ $\quad B\, 2\pi r = \mu_0 \frac{i_0}{\pi R^2} \pi r^2 ; \quad B = \frac{\mu_0 i_0}{2\pi R^2} r$

$r > R$ $\quad B\, 2\pi r = \mu_0 i_0 ; \quad B = \frac{\mu_0 i_0}{2\pi r}$

b) Find the magnetic flux through a rectangle that has one side of length W running down the center of the wire and another side of length R (see the figure)

$$\Phi = \int \vec{B} \cdot d\vec{S} = \int_0^R \frac{\mu_0 i_0 r}{2\pi R^2} W\, dr = \frac{\mu_0 i_0 W}{2\pi R^2} \frac{R^2}{2} = \boxed{\frac{\mu_0 i_0 W}{4\pi}}$$

c) Suppose that the wire is surrounded by a cylindrical conducting shell, inner radius a and outer radius b, with current i_0 flowing in the opposite direction. Find a force acting on a particle of charge q moving with the velocity V_0 parallel to the shell in the positive x direction. Explain your answer.



$r > c$: $\oint \vec{B} \cdot d\vec{r} = \mu_0 i$

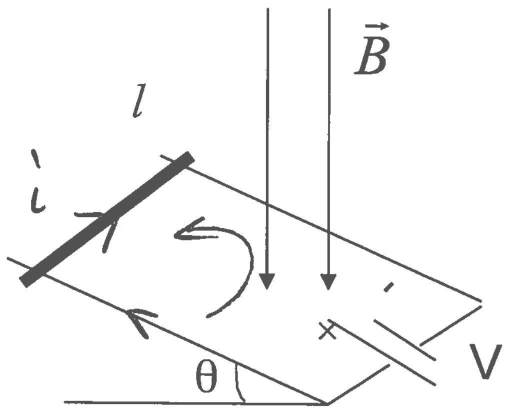
$B\, 2\pi r = \mu_0 (i_0 - i_0) = 0$

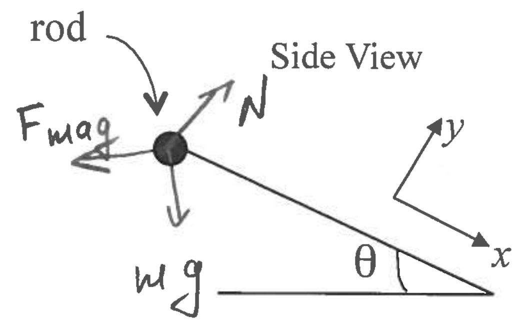
$B = 0 \quad \Rightarrow \boxed{F = 0}$

Problem 6: (15 points)

A rod of mass m and length l can slide without friction on a set of parallel rails which make an angle θ with the horizontal. The rod has resistance R; the rails and the connecting wire at the bottom are perfect conductors. The system is in a uniform magnetic field B directed as shown. The rod is released at time t=0. Ignore self-inductance.

(a) What should be the voltage V of the battery in order to keep the rod **at rest**? Draw the Free Body Diagram using the side view.





$V - iR = 0 \; ; \quad i = \frac{V}{R}$

$F_{mag} = i l B = \frac{V}{R} l B$

$F_x = m a_x$

$mg \sin\theta - F_{mag} \cos\theta = 0$

$mg \sin\theta - \frac{V}{R} l B \cos\theta = 0$

$\boxed{V = \frac{mgR}{lB} \tan\theta}$

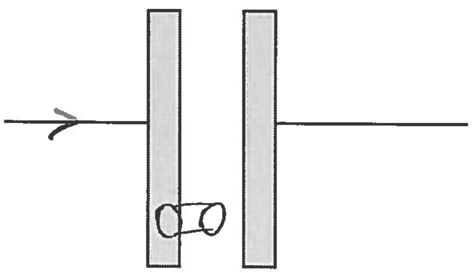
(b) Suppose the battery voltage is higher than the value found in (a). Find the direction of motion of the rod.

$\frac{V}{R} l B \cos\theta > mg \sin\theta$

The rod will move up

Problem 7: (13 points)

a) Consider a parallel plate capacitor in some circuit. The charge on a plate is changing according to $Q(t)=Q_0 \cos \omega t$, where Q_0 and ω are known constants. Find the current i into the plates and the displacement current. Calculate the displacement current through the flux of the electric field. Area of the plates is A.



$$i = \frac{dQ}{dt} = \boxed{-Q_0 \omega \sin \omega t}$$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E a = \frac{\sigma a}{\varepsilon_0}$$

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A \varepsilon_0}$$

$$i_D = \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S} = \varepsilon_0 \frac{d}{dt} \int \frac{Q}{A\varepsilon_0} \, ds = \varepsilon_0 \frac{d}{dt}\left(\frac{Q}{A\varepsilon_0} A\right) = \frac{dQ}{dt} = \boxed{-Q_0 \omega \sin \omega t}$$

b) Which of the following equations is the wave equation?

$$\frac{\partial E_y}{\partial x} = C \frac{\partial E_y}{\partial y} \qquad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad \frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

c) If we are given the electric and magnetic field that make up an electromagnetic wave,

$$E_y(x,t) = E_0 \sin(kx - \omega t) \qquad B_z(x,t) = B_0 \sin(kx - \omega t)$$

how must E_0 and B_0 be related? (Recall that Faraday's Law lead to a relation between the space derivative of $\vec{E}$ and the time derivative of $\vec{B}$.)

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$$E_0 k \cos(kx - \omega t) = -B_0(-\omega) \cos(kx - \omega t)$$

$$\boxed{\frac{E_0}{B_0} = \frac{\omega}{k} = c}$$