Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

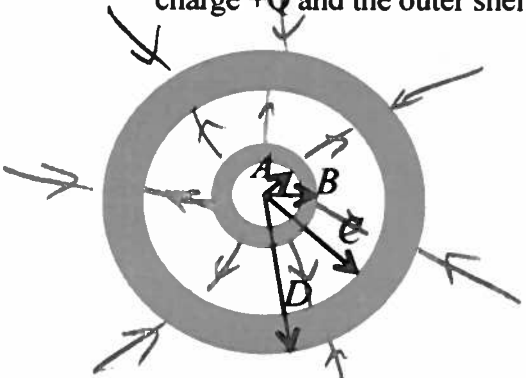
$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

Problem 2: (24 points)

a) A conducting spherical shell has inner radius A and outer radius B. It is concentric with another conducting spherical shell of inner radius C and outer radius D. The inner shell has charge +Q and the outer shell has charge -2Q.



$$\oint \vec{E}\cdot d\vec{s} = \frac{Q_{encl}}{\varepsilon_0}$$

Find the electric field at

i) $r < A$

$E = 0 \quad (Q_{encl} = 0)$

ii) $A < r < B$

$E = 0$ (conductor)

iii) $B < r < C$

$E\,4\pi r^2 = \frac{Q}{\varepsilon_0}$; $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ radially out

iv) $C < r < D$

$E = 0$ (conductor)

v) $r > D$

$E\,4\pi r^2 = \frac{1}{\varepsilon_0}(Q - 2Q) = -\frac{Q}{\varepsilon_0}$, $E = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ radially in

b) Sketch the electric field lines.

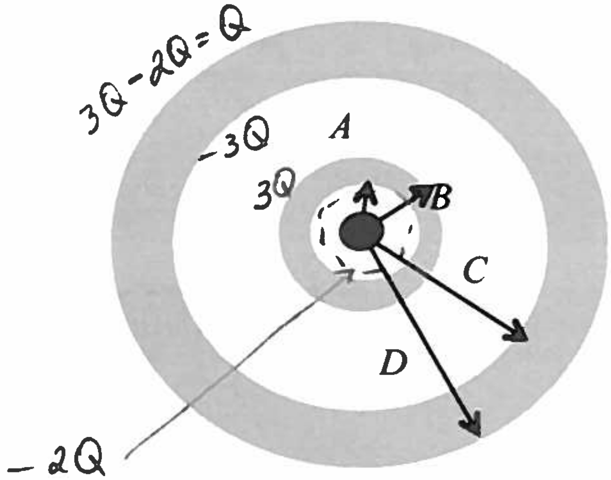
c) Find the difference in electric potential between $r = A$ and $r = \infty$: $V(\infty)$-$V(A)$.

$$V(\infty) - V(A) = -\int_A^\infty \vec{E}\cdot d\vec{r} = -\left[\int_A^B 0 + \int_B^C \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,dr + \right.$$

$$\left. + \int_C^D 0 + \int_D^\infty \left(-\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,dr\right)\right] = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\Big|_B^C - \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\Big|_D^\infty =$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{B} - \frac{1}{C} - \frac{1}{D}\right)$$

d) A conducting spherical shell has inner radius A and outer radius B. It is concentric with another conducting spherical shell of inner radius C and outer radius D. The inner shell has charge +Q and the outer shell has charge -2Q. A positive charge +2Q is placed at the center of the inner shell. Find the charge density σ at



i) $r = A$ $\qquad \sigma = \dfrac{-2Q}{4\pi A^2}$

ii) $r = B$ $\qquad \sigma = \dfrac{3Q}{4\pi B^2}$

iii) $r = C$ $\qquad \sigma = -\dfrac{3Q}{4\pi C^2}$

iv) $r = D$ $\qquad \sigma = \dfrac{Q}{4\pi D^2}$

e) A solid non-conducting sphere of radius R has charge spread throughout the volume so that the charge density is

$$\rho(r) = \rho_0 \frac{r}{R}$$

where ρ_0 is a known constant, r is a distance from the sphere's center.
Find the electric field everywhere.

$r < R$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E\, 4\pi r^2 = \frac{1}{\varepsilon_0} \int_0^r \rho \, dV =$$

$$= \frac{1}{\varepsilon_0} \int_0^r \rho_0 \frac{r}{R} 4\pi r^2 dr =$$

$$= \frac{1}{\varepsilon_0} \rho_0 \frac{4\pi}{R} \frac{r^4}{4} ; \quad E = \frac{\rho_0 r^2}{4 \varepsilon_0 R}$$

rad. out

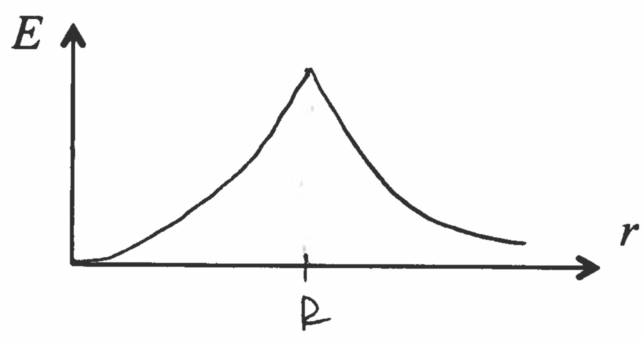
$r > R$

$$E\, 4\pi r^2 = \frac{1}{\varepsilon_0} \int_0^R \rho_0 \frac{r}{R} 4\pi r^2 dr =$$

$$= \frac{\rho_0}{\varepsilon_0 R} 4\pi \frac{R^4}{4}$$

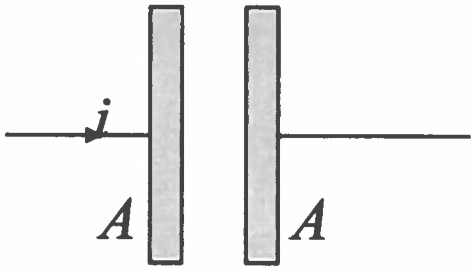
$$E = \frac{\rho_0 R^3}{4 \varepsilon_0} \frac{1}{r^2} \quad \text{rad. out}$$

Sketch the magnitude of electric field as a function of r.

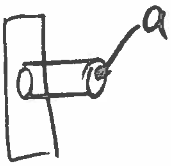


Problem 3: (12 points)

Consider two parallel plates of area A in some circuit:



The electric field between the plates is a function of time $E = E_0 \cos \omega t$. Find the current i in the wire and show that it is equal to the displacement current between the plates.



$$i = \frac{dQ}{dt}$$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

$$E a = \frac{Q}{A \varepsilon_0} a$$

$$E = \frac{Q}{A\varepsilon_0} ; \quad Q = EA\varepsilon_0 = A \varepsilon_0 E_0 \cos \omega t$$

$$i = \frac{dQ}{dt} = \boxed{-\omega A \varepsilon_0 E_0 \sin \omega t}$$

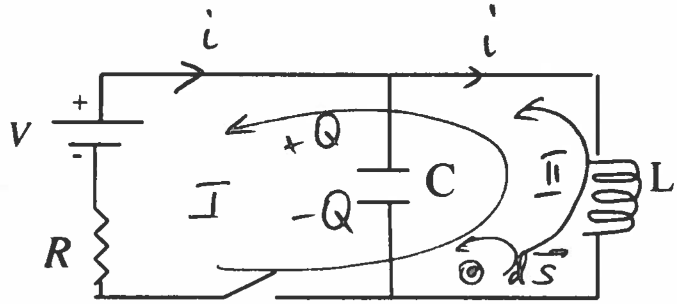
$$i_D = \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

$$\Phi_E = \int \vec{E} \cdot d\vec{S} = E_0 \cos \omega t \cdot A$$

$$i_D = \varepsilon_0 \frac{d}{dt} (A E_0 \cos \omega t) = \boxed{-\omega A \varepsilon_0 E_0 \sin \omega t}$$

Problem 4: (24 points)

a) In the circuit below the switch has been closed for a long time so it may be assumed that the steady state has been reached. Find the current through the resistor and the charges on the capacitor plates.



$$\oint \vec{E}\cdot d\vec{r} = 0$$

$$\mathrm{I}:\quad V - iR = 0$$

$$\boxed{i = \frac{V}{R}}$$

$$\mathrm{II}:\quad \frac{Q}{C} = 0 \Rightarrow \boxed{Q = 0}$$

b) At t=0 the switch is opened. Find the charge on the capacitor as a function of time. (Assume that L includes self-inductance).

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$i = -\frac{dQ}{dt}\ ; \quad \Phi = \pm Li = -Li$$

$$\frac{d\Phi}{dt} = -L\frac{di}{dt}$$

$$\frac{Q}{C} = L\frac{di}{dt} = -L\frac{d^2Q}{dt^2}$$

$$L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0$$

$$Q(t) = A\cos\omega t + B\sin\omega t$$

$$Q(t=0) = \boxed{A = 0}$$

$$i = -\frac{dQ}{dt} = A\omega\sin\omega t - B\omega\cos\omega t$$

$$i(t=0) = -B\omega = \frac{V}{R}$$

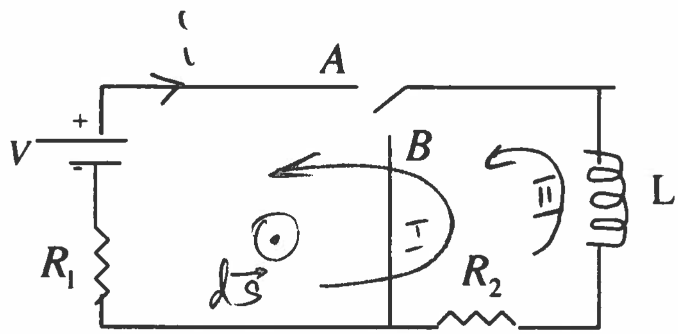
$$\boxed{B = -\frac{V}{R\omega}}$$

$$-L\cancel{B}\omega^2 \cancel{\sin\omega t} + \frac{1}{C}\cancel{B}\cancel{\sin\omega t} = 0$$

$$\boxed{\omega^2 = \frac{1}{LC}}$$

$$Q(t) = -\frac{V}{R\omega}\sin\omega t$$

c) In the circuit below the switch has been in the position A for a long time. Find the current through the resistor R_2.



$$\oint \vec{E}\cdot d\vec{r} = 0$$

$$V - i(R_1 + R_2) = 0$$

$$\boxed{i = \frac{V}{R_1 + R_2}}$$

I.

d) At $t=0$ the switch is moved to the position B. Find the current through R_2, as a function of time. (Assume that L includes self-inductance).

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$$

$$\Phi = \pm L i = -L i$$

$$\frac{d\Phi}{dt} = -L\frac{di}{dt}$$

$$-i_2 R = L\frac{di}{dt}$$

$$L\frac{di}{dt} + R_2 i = 0$$

$$\frac{di}{dt} + \frac{R_2}{L} i = 0$$

$$i(t) = \alpha e^{-\beta t}$$

$$\alpha(-\beta) e^{-\beta t} + \frac{R_2}{L}\alpha e^{-\beta t} = 0$$

$$\boxed{\beta = \frac{R_2}{L}}$$

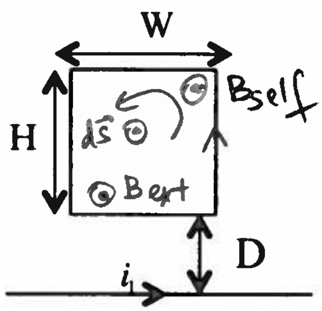
$$i(t) = \alpha e^{-\frac{R_2}{L}t}$$

$$i(t=0) = \alpha = \frac{V}{R_1 + R_2}$$

$$i(t) = \frac{V}{R_1 + R_2} e^{-\frac{R_2}{L}t}$$

Problem 5: (20 points)

A rectangular loop lies in the plane of the page. It has dimensions shown in the figure below. There is a long straight wire distance D from the loop with current $i_1(t) = i_0 e^{-\beta t}$ where i_0 and β are known constants. The wire from which the loop is made has resistivity ρ and cross sectional area a.



a) Find the direction of current in the loop.

Explain your answer within this box:
CCW i decreases -> B decreases => Φ_B decreases. Since $B_{external}$ is out, then B_{self} is out of page => i is CCW

b) Calculate the current in the loop as a function of time. Ignore self-inductance.

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 i_1 \Rightarrow B = \frac{\mu_0 i_0 e^{-\beta t}}{2\pi r}; \quad \boxed{R = \frac{2\rho(W+H)}{a}}$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$$

$$\Phi = \int \vec{B}\cdot d\vec{s} = \int_D^{D+H} \frac{\mu_0 i_0 e^{-\beta t}}{2\pi r} W\,dr = \frac{\mu_0 i_0 e^{-\beta t} W}{2\pi} \ln\frac{D+H}{D}$$

$$\frac{d\Phi}{dt} = \frac{\mu_0 i_0 W}{2\pi} \ln\frac{D+H}{D}(-\beta) e^{-\beta t}$$

$$iR = \frac{\mu_0 i_0 W \beta}{2\pi} \ln\frac{D+H}{D} e^{-\beta t}; \quad \boxed{i = \frac{\mu_0 i_0 W \beta}{2\pi R} \ln\frac{D+H}{D} e^{-\beta t}}$$

c) Derive the equation describing the current in the loop if self-inductance of the loop is L. Do not solve it.

$$\Phi_{self} = Li; \quad \frac{d\Phi}{dt} = L\frac{di}{dt}; \quad iR = \frac{\mu_0 i_0 W \beta}{2\pi} \ln\frac{D+H}{D} e^{-\beta t} - L\frac{di}{dt}$$

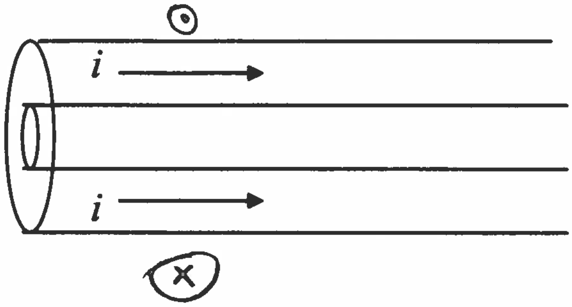
d) Bonus (3 points) Calculate the total energy dissipated in the loop from $t = 0$ to $t = \infty$. Ignore self-inductance.

$$\text{Energy} = \int_0^\infty i^2 R\,dt = \frac{(\mu_0 i_0 W \beta)^2}{(2\pi R)^2} \ln^2\frac{D+H}{D} R \int_0^\infty e^{-2\beta t}\,dt =$$

$$= \frac{(\mu_0 i_0 W \beta)^2}{8\pi^2 \beta R} \ln^2\frac{D+H}{D}$$

Problem 6: (20 points)

An infinitely long, hollow cylindrical wire has inner radius A and outer radius B. A current i is uniformly distributed over its cross-section. a) Find the magnetic field everywhere (r<A; A<r<B; r>B)



1) $r<A \quad \oint \vec{B}\cdot d\vec{r} = \mu_0 i; \quad B = 0$

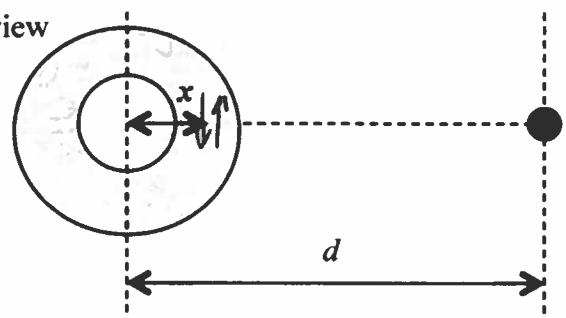
2) $A<r<B \quad B 2\pi r = \mu_0 \frac{i}{\pi B^2 - \pi A^2}(\pi r^2 - \pi A^2)$

$$B = \frac{\mu_0 i}{2\pi(B^2-A^2)} \frac{r^2-A^2}{r}$$

3) $B 2\pi r = \mu_0 i, \quad B = \frac{\mu_0 i}{2\pi r}$

b) In addition to the hollow wire described above, there is a thin infinitely long wire positioned at a distance d from the center of the hollow wire. Both wires have currents i pointed into the page. Find the net magnetic field at an arbitrary point $A < x < B$ on the line connecting the centers of the wires.

Side view



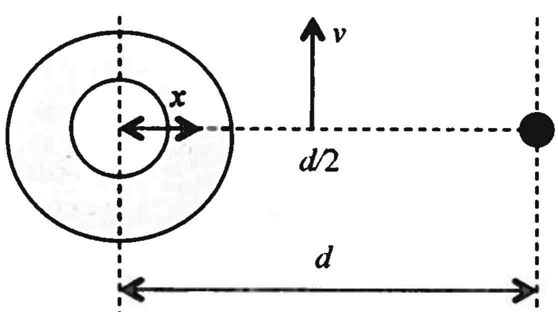
$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

$$\vec{B}_1 = \frac{\mu_0 i}{2\pi(B^2-A^2)} \frac{r^2-A^2}{r} \downarrow$$

$$\vec{B}_2 = \frac{\mu_0 i}{2\pi(d-x)} \uparrow$$

$$\vec{B} = \frac{\mu_0 i}{2\pi}\left(\frac{r^2-A^2}{B^2-A^2}\frac{1}{r} - \frac{1}{d-x}\right) \downarrow$$

c) There is a particle of positive charge q moving with velocity of magnitude v vertically up at the point located at the point $x = d/2$ in the middle of the line connecting the centers of the wires ($d/2 > B$). Find the force acting on the particle at this point.



$$\vec{F} = q\vec{v}\times\vec{B}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = 0$$

or $\vec{v} \parallel \vec{B}$