Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

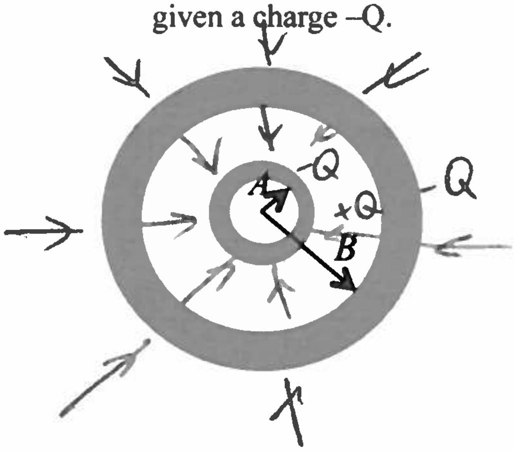
$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot dr = \mu_0 \left(i + \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S} \right)$$

Problem 2: (24 points)

a) A conducting spherical shell has inner radius A and thickness T. There is a larger concentric spherical conducting shell with inner radius B and thickness T. The inner shell is given a charge –Q.



a) Find the charge per unit area on all surfaces.

$$\boxed{\begin{array}{ll} r = A & \sigma = 0 \\ r = A+T & \sigma = -\dfrac{Q}{4\pi(A+T)^2} \\ r = B & \sigma = \dfrac{Q}{4\pi B^2} \\ r = B+T & \sigma = -\dfrac{Q}{4\pi(B+T)^2} \end{array}}$$

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

b) Find the electric field at

i) $r < A$

$\vec{E} = 0 \; (Q_{encl} = 0)$

ii) $A < r < A+T$

$\vec{E} = 0$ (conductor)

iii) $A+T < r < B$

$$E\, 4\pi r^2 = -\frac{Q}{\varepsilon_0}, \quad \vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{i}_r$$

iv) $B < r < B+T$

$\vec{E} = 0$ (conductor)

v) $r > B+T$

$$E\, 4\pi r^2 = -\frac{Q}{\varepsilon_0}; \quad \vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{i}_r$$

c) Sketch the electric field lines.

d) Find the difference in electric potential between $r = A$ and $r = B$: $V(B)-V(A)$.

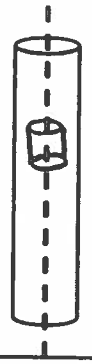
$$V(B) - V(A) = -\int_A^B \vec{E}\cdot d\vec{r} = +\int_{A+T}^{B} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,dr =$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\Bigg|_{A+T}^{B} = -\frac{1}{4\pi\varepsilon_0} Q\left(\frac{1}{B} - \frac{1}{A+T}\right)$$

Result: $V(B)-V(A) = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{A+T} - \dfrac{1}{B}\right)$

e) A very, very long insulating cylinder of radius A and length L has a charge Q uniformly spread throughout its volume. Consider only points very far from the ends so that the cylinder can be assumed to be infinitely long.

i) Find the electric field as a function of r, the distance from the axis of the cylinder, for all values of r.



$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

1) $r < A$

$$E\, 2\pi r \ell = \frac{1}{\varepsilon_0}\frac{Q}{\pi A^2 L}\pi r^2 \ell$$

$$E = \frac{Q}{2\pi\varepsilon_0 A^2 L}\, r \quad \text{radially out}$$

2) $r > A$

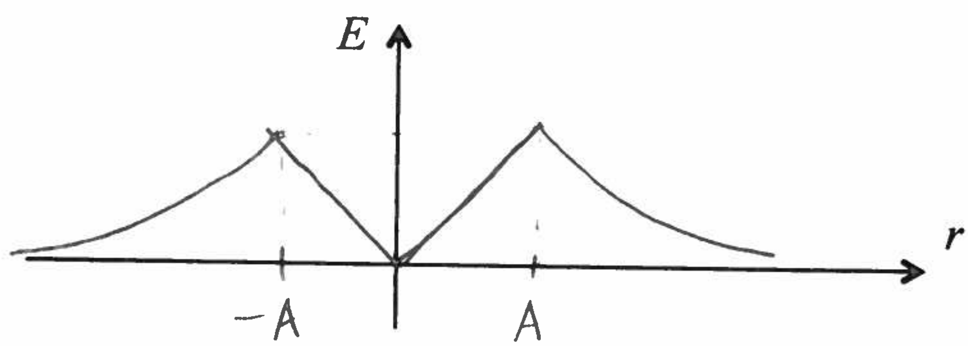
$$E\, 2\pi r \ell = \frac{1}{\varepsilon_0}\frac{Q}{\pi A^2 L}\pi A^2 \ell$$

$$E = \frac{1}{2\pi\varepsilon_0 L}\, Q\, \frac{1}{r} \quad \text{radially out}$$

Result: $r < A \quad E = \frac{Q}{2\pi\varepsilon_0 A^2 L}\, r$

$r > A \quad E = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}$ radially out

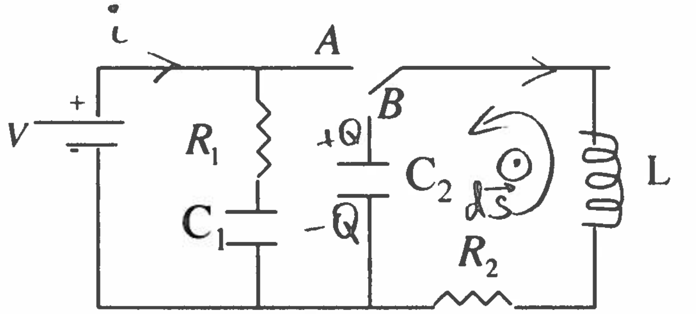
ii) Sketch the magnitude of electric field as a function of r.



Problem 3: (21 points)

In the circuit below the switch has been in the position A for a long time.

a) Find the current through the resistor R_2.



$$\oint \vec{E}\cdot d\vec{r} = 0$$

$$V - iR = 0$$

$$\boxed{i = \frac{V}{R_2}}$$

b) At t=0 the switch is moved to the position B. Starting from some famous law, derive the equation that can be solved to find the charges on the capacitor C_2 and current through R_2, as a function of time.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\underbrace{\int \vec{B}\cdot d\vec{S}}_{\Phi}$$

$$\Phi = -Li\ ; \quad \frac{d\Phi}{dt} = -L\frac{di}{dt}\ ; \quad i = -\frac{dQ}{dt}$$

$$\frac{Q}{C_2} - iR_2 = L\frac{di}{dt}\ ; \quad \frac{Q}{C_2} + R_2\frac{dQ}{dt} = -L\frac{d^2Q}{dt^2}$$

$$\boxed{L\frac{d^2Q}{dt^2} + R_2\frac{dQ}{dt} + \frac{1}{C_2}Q = 0}$$

c) Replace C_2 by a perfectly conducting wire. Find the current through R_2, as a function of time.

$$L\frac{di}{dt} + R_2 i = 0$$

$$i(t) = \alpha e^{-\beta t}$$

$$L\alpha(-\beta)e^{-\beta t} + R_2\alpha e^{-\beta t} = 0$$

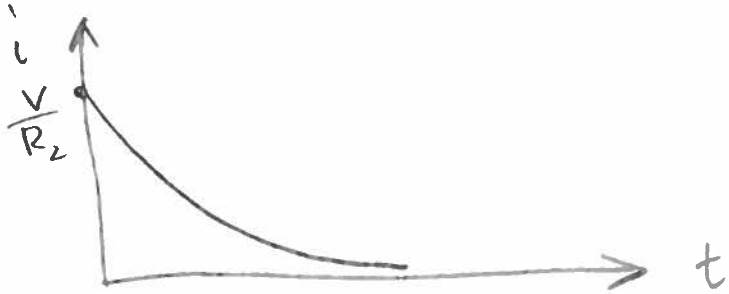
$$\boxed{\beta = \frac{R_2}{L}}$$

$$i(t) = \alpha e^{-\frac{R_2}{L}t}$$

$$i(t=0) = \alpha = \frac{V}{R_2}$$

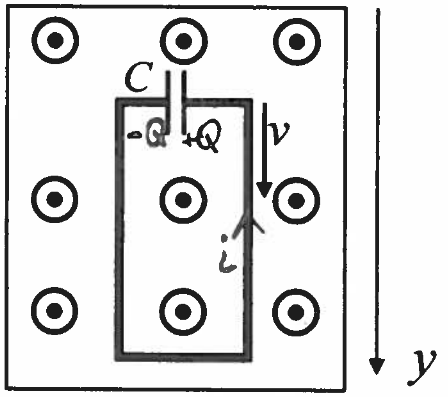
$$\boxed{i(t) = \frac{V}{R_2}e^{-\frac{R_2}{L}t}}$$

d) Plot the current as a function of time schematically.



Problem 4: (22 points)

A vertically oriented loop with capacitance C, resistance R, length l and width w, falls from a region where the magnetic field of a given magnitude B is horizontal, uniform, and pointing out of the page as shown in the Figure. At $t = 0$ the capacitor was uncharged. Ignore self-inductance. Note that the positive y direction is down.



a) Find the charges on the capacitor plates and current in the loop as a function of the velocity v

if the loop is falling down but is completely within the region having magnetic field B.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\underbrace{\int \vec{B}\cdot d\vec{S}}_{\Phi}$$

$$\frac{d\Phi}{dt} = 0 \Rightarrow \boxed{i = 0;\ Q = 0}$$

b) Once the lower segment leaves the region with magnetic field, find the direction of the current in the loop. Explain your answer within this box:

area decreases, flux decreases, B_{self} ⊙
to oppose the change
i is in ccw direction

c) Find the equation that can be solved to find the charge on the capacitor as a function of the velocity of the loop v once the lower segment leaves the region with magnetic field.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$\Phi = \int \vec{B}\cdot d\vec{S} = Bwy$$

$$\frac{d\Phi}{dt} = Bw\frac{dy}{dt} = -Bwv \qquad \frac{dy}{dt} = -v$$

$$\frac{Q}{C} + iR = Bwv; \quad i = \frac{dQ}{dt}$$

$$\frac{1}{C}Q + R\frac{dQ}{dt} = Bwv$$

$$\boxed{R\frac{dQ}{dt} + \frac{1}{C}Q = Bwv}$$

d) Solve for the charge Q and current in the loop.

$$Q(t) = \underset{(1)}{Q_{particular}} + \underset{(2)}{Q_{homogeneous}}$$

(1): $Q(t) = BwvC$

(2): $Q(t) = \alpha e^{-\beta t}$

$$R(-\beta)\alpha e^{-\beta t} + \frac{1}{C}\alpha e^{-\beta t} = 0$$

$$\beta = \frac{1}{RC}$$

$$Q(t) = BwvC + \alpha e^{-\frac{t}{RC}}$$

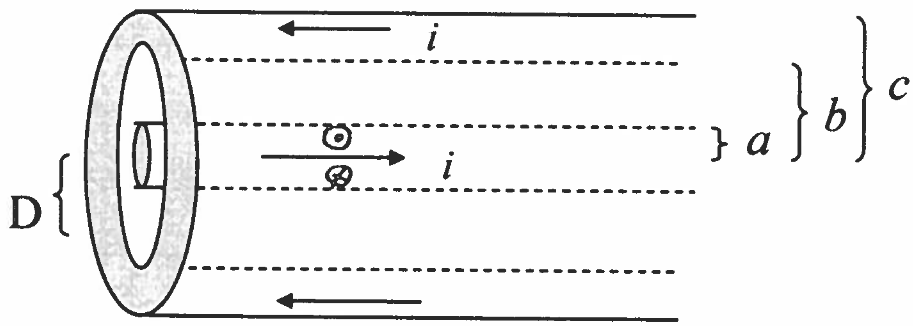
$$Q(t=0) = BwvC + \alpha = 0$$

$$\alpha = -BwvC$$

$$\boxed{Q(t) = BwvC\left(1 - e^{-\frac{t}{RC}}\right)}$$

Problem 5: (21 points)

a) Coaxial cable: Consider an infinitely long cylindrical conductor carrying a current ***i*** spread uniformly over its cross section and a cylindrical conducting shell around it with a current i flowing in the opposite direction. It is uniformly spread over the cross section of the shell.



Find the magnetic field at

$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$

i) $r < a$

curren into th page

$B\, 2\pi r = \mu_0 \frac{i}{\pi a^2} \pi r^2$; $B = \frac{\mu_0 i}{2\pi a^2} r$

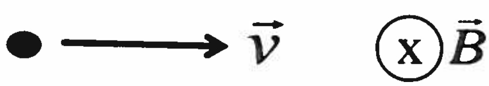
(V) B

ii) $r > c$

$B\, 2\pi r = i - i = 0$

$B = 0$

b) Velocity selector: A proton moves with constant velocity v to the right through a region where there is a uniform magnetic field of magnitude B that points into the page. There is also an electric field in this region.



$q v B$

$q E$

i) What is the direction of the electric field ? down

ii) What is the magnitude of the electric field ? $\Sigma \vec{F} = 0$; $qE = qvB$; $E = vB$

c) How would your answer in part b) change if there were an electron instead of a proton?

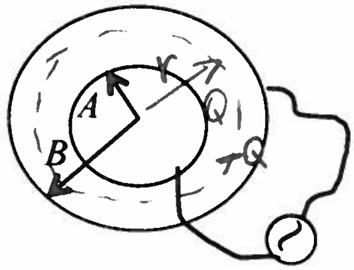
i) What is the direction of the electric field ? down

ii) What is the magnitude of the electric field ? the same

$E = vB$

Problem 6: (12 points)

A spherical capacitor is connected to a generator. The voltage between the capacitor's plates is changing according to $V(t) = V_0 \cos\omega t$. Find the displacement current through a sphere of an arbitrary radius r, $A < r < B$; radii A and B are given.



$$i_D = \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$\Phi = \int \vec{E} \cdot d\vec{S} \equiv \oint_{\substack{\text{sphere} \\ \text{radius } r}} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q$$

$$\frac{d\Phi}{dt} = \frac{1}{\varepsilon_0} \frac{dQ}{dt} \underset{Q = CV}{=} \frac{1}{\varepsilon_0} \frac{d}{dt} CV = \frac{1}{\varepsilon_0} C \frac{dV}{dt} =$$

$$= -\frac{1}{\varepsilon_0} C V_0 \omega \sin\omega t$$

$$C = ? \quad C = \frac{Q}{|\Delta V|}; \quad V(B) - V(A) = -\int_A^B \vec{E} \cdot d\vec{r};$$

$$E 4\pi r^2 = \frac{Q}{\varepsilon_0}; \quad E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$V(B) - V(A) = -\int_A^B \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} dr = \frac{1}{4\pi\varepsilon_0} Q \left(\frac{1}{B} - \frac{1}{A}\right)$$

$$C = \frac{Q}{|\Delta V|} = \frac{4\pi\varepsilon_0}{\frac{1}{A} - \frac{1}{B}}; \quad \boxed{i_D = -\frac{4\pi\varepsilon_0}{\frac{1}{A} - \frac{1}{B}} V_0 \omega \sin\omega t}$$