Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

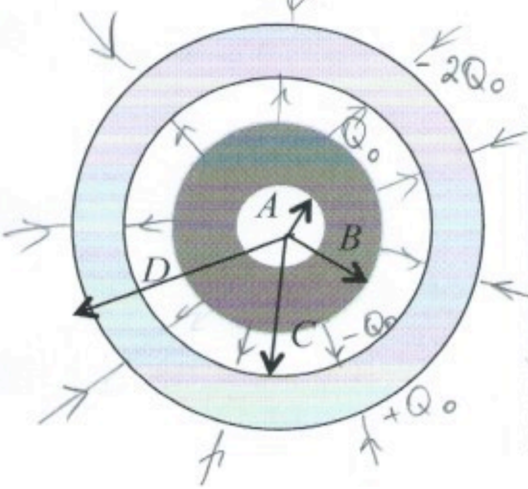
$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \left(i + \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S} \right)$$

Problem 2: (20 points)

A spherical conducting shell, inner radius A and outer radius B, is charged with charge Q_0. It is surrounded by a conducting spherical shell of inner radius C and outer radius D, which is charged with charge $-2Q_0$.



a) Find the charge per unit area on all surfaces.

$r=A \quad \sigma = 0$ $r=B \quad \sigma = \frac{Q_0}{4\pi B^2}$ $r=C \quad \sigma = -\frac{Q_0}{4\pi C^2}$	$r=D \quad \sigma = -\frac{Q_0}{4\pi D^2}$

b) Find the electric field at

i) $r < A$

$E = 0 \ (Q_{encl} = 0)$

ii) $A < r < B$

$E = 0$ (conductor)

iii) $B < r < C$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}; \quad E\, 4\pi r^2 = \frac{Q_0}{\varepsilon_0}; \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \text{ rad. out}$$

iv) $C < r < D$

$E = 0$ (conductor)

v) $r > D$ $\quad E\, 4\pi r^2 = \frac{-Q_0}{\varepsilon_0}; \quad \vec{E} = -\frac{Q_0}{4\pi\varepsilon_0 r^2}$ rad out

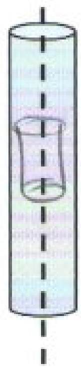
c) Sketch the electric field lines.

d) Find the difference in electric potential between $r = 0$ and $r = \infty$, $V(\infty)$-$V(0)$.

$$V(\infty) - V(0) = -\int_0^\infty \vec{E} \cdot d\vec{r} = -\left[\int_0^A 0 + \int_A^B 0 + \int_B^C \frac{Q_0}{4\pi\varepsilon_0 r^2}\, dr + \int_C^D 0 + \int_D^\infty \frac{-Q_0}{4\pi\varepsilon_0}\frac{dr}{r^2}\right.$$

$$= -\left[-\frac{Q_0}{4\pi\varepsilon_0}\frac{1}{r}\Big|_B^C + \frac{Q_0}{4\pi\varepsilon_0}\frac{1}{r}\Big|_D^\infty\right] = \frac{Q_0}{4\pi\varepsilon_0}\left[\frac{1}{C} - \frac{1}{B} + \frac{1}{D}\right]$$

d) A long, nonconducting, solid cylinder of radius R has a nonuniform volume charge density $\rho = cr^2$.



a) Find the electric field at

1) $r < R$

$$V = \pi r^2 \ell$$

$$\frac{dV}{dr} = 2\pi r \ell$$

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E\, 2\pi r \ell = \frac{1}{\varepsilon_0} \int \rho\, dV = \frac{1}{\varepsilon_0} \int_0^r c r^2 2\pi r \ell\, dr$$

$$E\, \cancel{2\pi} r \cancel{\ell} = \frac{1}{\varepsilon_0} c \cancel{2\pi} \cancel{\ell} \frac{r^4}{4}; \quad E = \frac{c}{4\varepsilon_0} r^3 \text{ rad. out}$$

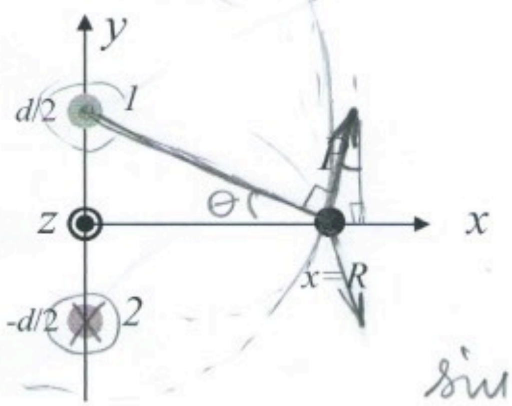
2) $r > R$

$$E\, \cancel{2\pi} r \cancel{\ell} = \frac{1}{\varepsilon_0} \int_0^R c r^2 2\pi r \ell\, dr = \frac{c}{\varepsilon_0} \cancel{2\pi} \cancel{\ell} \frac{R^4}{4}$$

$$E = \frac{cR^4}{4\varepsilon_0} \frac{1}{r} \text{ rad. out}$$

Problem 3: (14 points)

a) Two very long parallel wires are separated by distance d (see the figure below). Current in wire 1 has magnitude i and is out of the page. . Current in wire 2 has the same magnitude i and is into the page. In unit-vector notation, what is the net magnetic field at point P at distance R due to the two currents?



$\oint \vec{B}\cdot d\vec{r} = \mu_0 i$; $B = \frac{\mu_0 i}{2\pi r}$; $r = \sqrt{R^2 + \ldots}$

$\Sigma B_y = 0$ from symmetry

$B_x = \frac{2\mu_0 i}{2\pi r} \sin\theta$

$$\sin\theta = \frac{d/2}{r} = \frac{d/2}{\sqrt{R^2 + \frac{d^2}{4}}}$$

$$B_x = \frac{\mu_0 i d}{2\pi\left(R^2 + \frac{d^2}{4}\right)}$$

$$\vec{B} = B_x \vec{i}_x + 0\,\vec{i}_y$$

b) A negatively charged particle is injected at point P with a velocity $\vec{v} = v_0\vec{i}_y$, where v_0 is a constant. What constant electric field (magnitude and direction) would have to be applied for the particle to experience no net force? Ignore gravity.

$\uparrow v$

$\odot \rightarrow B$

$qvB - qE = 0$

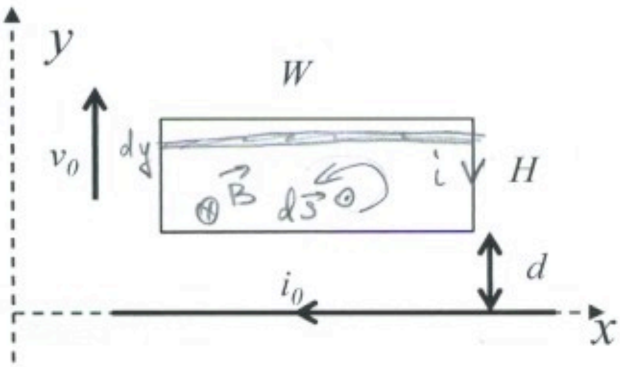
$\boxed{E = vB}$ $\boxed{\vec{E} = vB\vec{i}_z}$

c) A positively charged particle is injected at point P with a velocity $\vec{v} = v_0\vec{i}_y$, where v_0 is a constant. What constant electric field (magnitude and direction) would have to be applied for the particle to experience no net force? Ignore gravity.

$\boxed{E = vB}$ $\otimes F_{mag}$ $\odot qE$ $\boxed{\vec{E} = vB\vec{i}_z}$

Problem 4: (18 points)

At $t = 0$ a rectangular loop of wire with length W, width H, and resistance R is located at distance d from an infinitely long wire carrying current i_0. The loop is moved away from the wire at constant speed v_0.



a) Find the direction of the current in the loop. Explain your answer within this box:

B decreases, Φ_B decreases, B_{self} is in the same direction as $B_{original}$, $i_{induced}$ is CW

b) Find the current induced in the loop as a function of time. Ignore self-inductance.

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 i \,;\quad \vec{B} = \frac{\mu_0 i_0}{2\pi y}\otimes \,;\quad \oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

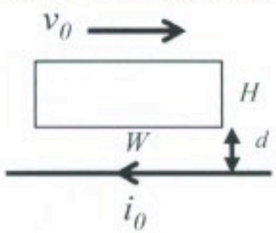
$$\Phi = \int \vec{B}\cdot d\vec{S} = -\int_y^{y+H} \frac{\mu_0 i_0}{2\pi y} W\,dy = -\frac{\mu_0 i_0}{2\pi} W\left(\ln(y+H) - \ln y\right)$$

$$\frac{d\Phi}{dt} = -\frac{\mu_0 i_0 W}{2\pi}\left(\frac{1}{y+H}\underbrace{\frac{dy}{dt}}_{=v_0} - \frac{1}{y}\underbrace{\frac{dy}{dt}}_{=v_0}\right);\quad y = d + v_0 t$$

$$-iR = \frac{\mu_0 i_0 W}{2\pi}\left(\frac{1}{y+H} - \frac{1}{y}\right) v_0$$

$$i = \frac{\mu_0 i_0 W}{2\pi R}\left(\frac{1}{y} - \frac{1}{y+H}\right) v_0 \quad \text{CW}$$

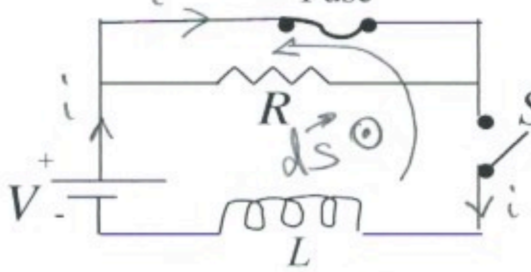
b) Find the current induced in the loop if the loop is moving along the wire. Ignore self-inductance.



$$i = 0 \qquad \frac{d\Phi_B}{dt} = 0$$

Problem 5: (18 points)

In the circuit below the fuse has zero resistance as long as the current through it remains less than i_0. If the current reaches i_0, the fuse "blows" and thereafter has infinite resistance. Switch S is closed at $t=0$.



a) Find the current as a function of time.

R is shorted out

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial \Phi}{\partial t}$$

$$\Phi = -Li \,;\quad \frac{d\Phi}{dt} = -L\frac{di}{dt}$$

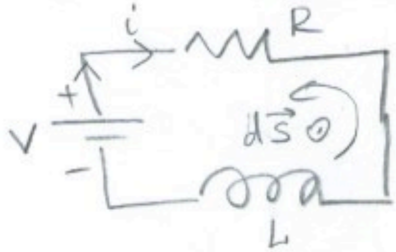
$$V = L\frac{di}{dt}\,;\quad \frac{di}{dt} = \frac{V}{L}\,;\quad i = \int \frac{V}{L}dt = \frac{V}{L}t + \underset{}{Const} \quad = 0 \; (i(t=0)$$

$$\boxed{i = \frac{V}{L}t}$$

b) Find time t_1 when the fuse blows.

$$\frac{V}{L}t_1 = i_0 \implies \boxed{t_1 = \frac{L i_0}{V}}$$

c) Redefine the moment of time when the fuse blows as $t=0$. The current in the circuit at $t=0$ is i_0. Find the current in the circuit as a function of time.



$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$\Phi = -Li \,;\quad \frac{d\Phi}{dt} = -L\frac{di}{dt}$$

$$V - iR = L\frac{di}{dt}$$

$$L\frac{di}{dt} + Ri = V$$

$$i = i_{ss} + i_h$$

$$i_{ss} = \frac{V}{R}$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

$$i_h(t) = \alpha e^{-\beta t}$$

$$\alpha(-\beta)e^{-\beta t} + \frac{R}{L}\alpha e^{-\beta t} = 0$$

$$\beta = \frac{R}{L}$$

$$i(t) = \frac{V}{R} + \alpha e^{-\frac{R}{L}t}$$

$$i(t=0) = \frac{V}{R} + \alpha = i_0$$

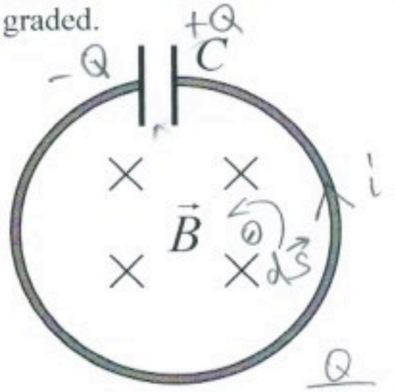
$$\alpha = i_0 - \frac{V}{R}$$

$$i(t) = \frac{V}{R} + \left(i_0 - \frac{V}{R}\right)e^{-\frac{R}{L}t}$$

Problem 6: (20 points)

A single loop of wire with an area A is placed in the time varying magnetic field directed into the page. The magnitude of the magnetic field is given by $B=B_0+\alpha t$, where B_0 and α are constants and t is time. The loop has a capacitor, capacitance C that was initially uncharged. **It also has resistance *R*.**

a) Starting from some famous law, derived the equation that could be solved to find the charges on the capacitor as a function of time if **the self-inductance of the loop is *L***. Do not solve it. Please note that without a direction of current and charges on the capacitor indicated on the circuit the problem will not be graded.



$$\oint \vec{B}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$\Phi_{ext} = \int \vec{B}_{ext}\cdot d\vec{S} = -(B_0+\alpha t)A$$

$$\frac{d\Phi_{ext}}{dt} = -\alpha A;\ \Phi_{self} = Li;\ \frac{d\Phi_{self}}{dt} = L\frac{di}{dt}$$

$$\frac{Q}{C} + iR = -\left[-\alpha A + L\frac{di}{dt}\right] = \alpha A - L\frac{di}{dt}$$

$$i = \frac{dQ}{dt};\ \frac{Q}{C} + R\frac{dQ}{dt} = \alpha A - L\frac{d^2Q}{dt^2};\ \boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \alpha A}$$

b) **Ignore the self-inductance**. Find the charges on the capacitor as a function of time.

$$R\frac{dQ}{dt} + \frac{1}{C}Q = \alpha A$$

$$Q(t) = Q_{ss} + Q_h$$

$$Q_{ss} = C\alpha A$$

$$Q_h = d e^{-\beta t}$$

$$R d(-\beta) e^{-\beta t} + \frac{1}{C} d e^{-\beta t} = 0$$

$$\beta = \frac{1}{RC}$$

$$Q(t) = C\alpha A + d e^{-\beta t}$$

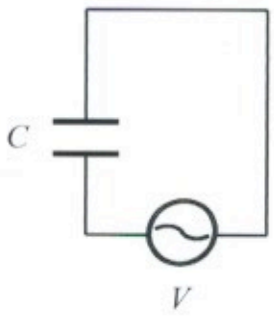
$$Q(t=0) = C\alpha A + d = 0$$

$$d = -C\alpha A$$

$$\boxed{Q(t) = C\alpha A\left(1 - e^{-\frac{1}{RC}t}\right)}$$

Problem 7: (7 points)

A capacitor is connected to a battery with $V = V_m \sin \omega t$, V_m and ω are given constants. The maximum value of the displacement current is I_d. Ignore the resistance and self-inductance of the circuit.



a) What is the maximum value of the current i in the circuit?

$$\max i = \max i_D = I_D$$

b) Find the displacement current between the plates, i_d as a function of time if the distance between the plates is d and the area of the plates is A.

$$i_d = \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

$$\Phi = \int \vec{E} \cdot d\vec{S} = \frac{V}{d} A = \frac{V_m}{d} \sin \omega t \, A$$

$$\oint \vec{E} \cdot d\vec{r} = -[V(\vec{r}_2) - V(\vec{r}_1)]$$

$$E d = \Delta V, \quad E = \frac{\Delta V}{d}$$

$$\frac{d\Phi}{dt} = \frac{V_m}{d} A \omega \cos \omega t$$

$$\boxed{i_d = \varepsilon_0 \frac{A}{d} V_m \omega \cos \omega t}$$