Problem 1: (5 points)

Write Maxwell's equations in the integral form.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon_0}$$

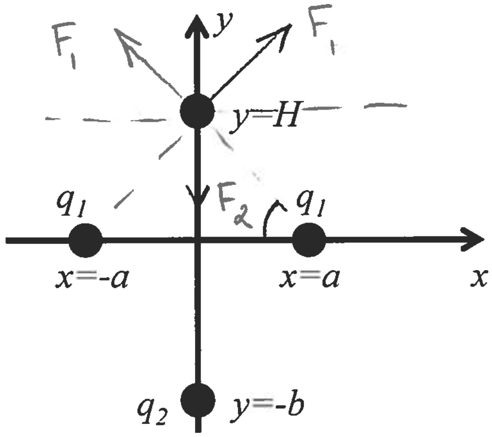
$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \left(i + \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S} \right)$$

Problem 2: (15 points)

Three charges are placed as shown. The distances a and b are known. The charges q_1 on the x axis are known and positive. The charge q_2 at $y = -b$ is unknown. What must be the unknown charge q_2 if the electric field is to be zero at x=0, y=H? Here H is known and positive.



q_2 is negative

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$E_x = 0$ from symmetry

$$E_y = \frac{1}{4\pi\varepsilon_0}\left(2\frac{q_1}{a^2+H^2}\sin\theta - \frac{|q_2|}{(b+H)^2}\right) = 0$$

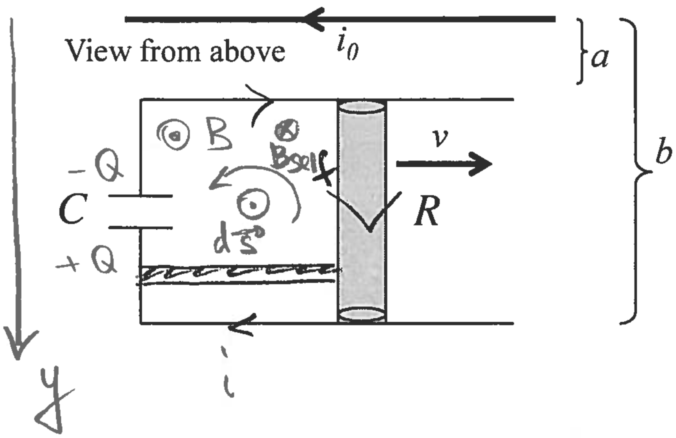
$$\sin\theta = \frac{H}{\sqrt{a^2+H^2}}$$

$$2\frac{q_1 H}{(a^2+H^2)^{3/2}} - \frac{|q_2|}{(b+H)^2} = 0$$

$$q_2 = \frac{2q_1 H(b+H)^2}{(a^2+H^2)^{3/2}}$$ negative

Problem 3: (20 points)

A rod with resistance R slides without friction on two resistance-free tracks which are connected by a capacitor, C. A wire, carrying a constant current i_0, is parallel to the tracks, in the same plane.



a) Find the direction of the current in the loop. Explain your answer within this box:

area $\uparrow$, $\Phi \uparrow$, $B_{self} \uparrow \downarrow B_{ext}$

$\Rightarrow i$ is CW

b) Ignore self-inductance L. Find the charge on the capacitor as a function of time assuming that at t=0 the capacitor was uncharged.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 i$$

$$B\, 2\pi r = \mu_0 i_0 ; \quad B = \frac{\mu_0 i_0}{2\pi r} \odot \text{ (below the wire with } i_0)$$

$$\Phi = \int \vec{B}\cdot d\vec{S} = \int_a^b \frac{\mu_0 i_0}{2\pi y}\, x\, dy = \frac{\mu_0 i_0 x}{2\pi} \ln\frac{b}{a}$$

$$\frac{d\Phi}{dt} = \frac{\mu_0 i_0 v}{2\pi} \ln\frac{b}{a}$$

$$-\frac{Q}{C} - iR = -\frac{\mu_0 i_0 v}{2\pi} \ln\frac{b}{a}$$

$$i = \frac{dQ}{dt}$$

$$\boxed{R\frac{dQ}{dt} + \frac{1}{C}Q = \frac{\mu_0 i_0 v}{2\pi} \ln\frac{b}{a}}$$

$$Q(t) = Q_{ss} + Q_h$$

$$Q_{ss} = c\,\frac{\mu_0 i_0 v}{2\pi} \ln \frac{b}{a}$$

$$R\,\frac{dQ_h}{dt} + \frac{1}{c}\,Q_h = 0$$

$$Q_h = \alpha\, e^{-\beta t}$$

$$R\alpha(-\beta)\, e^{-\beta t} + \frac{1}{c}\, e^{-\beta t} = 0$$

$$\beta = \frac{1}{RC}$$

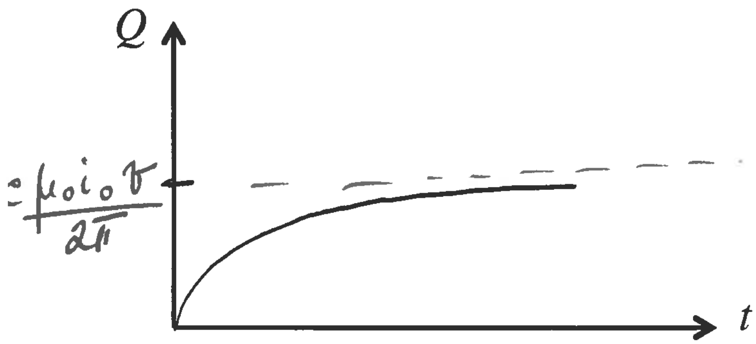
$$Q(t) = c\,\frac{\mu_0 i_0 v}{2\pi} \ln \frac{b}{a} + \alpha\, e^{-\frac{t}{RC}}$$

$$Q(t=0) = c\,\frac{\mu_0 i_0 v}{2\pi} \ln \frac{b}{a} + \alpha = 0$$

$$\alpha = -\frac{c\,\mu_0 i_0 v}{2\pi} \ln \frac{b}{a}$$

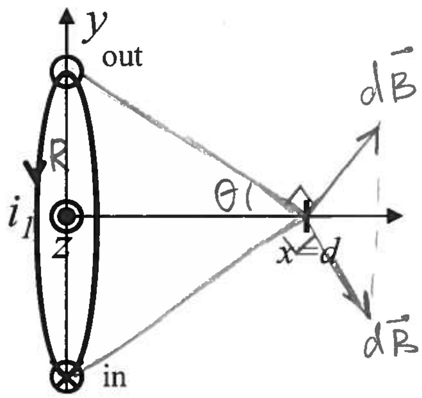
$$Q(t) = c\,\frac{\mu_0 i_0 v}{2\pi} \ln \frac{b}{a}\left(1 - e^{-\frac{t}{RC}}\right)$$

c) Plot the charge as a function of time schematically.



Problem 4: (18 points)

a) A loop with current i_1 has radius R. Find the magnetic field created by this current at $x=d$ (see the figure below).



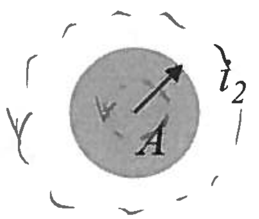
$$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3}; \quad B_y = 0 \text{ from symmetry}$$

$$dB_x = \frac{\mu_0 i_1}{4\pi}\frac{ds}{r^2}\sin\theta; \quad \sin\theta = \frac{R}{r}; \quad r = \sqrt{R^2+d^2}$$

$$B_x = \frac{\mu_0 i_1}{4\pi}\frac{R}{(R^2+d^2)^{3/2}}\int ds =$$

$$= \frac{\mu_0 i_1}{4\pi}\frac{R}{(R^2+d^2)^{3/2}}\,2\pi R = \frac{\mu_0 i_1 R^2}{2(R^2+d^2)^{3/2}}$$

b) An infinitely long wire of radius A has current i_2 out of the page. Find the magnetic field everywhere: at $r<A$ and $r>A$.



$$\oint \vec{B}\cdot d\vec{r} = \mu_0 i$$

$$B\,2\pi r = \mu_0 \frac{i_2}{\pi A^2}\pi r^2$$

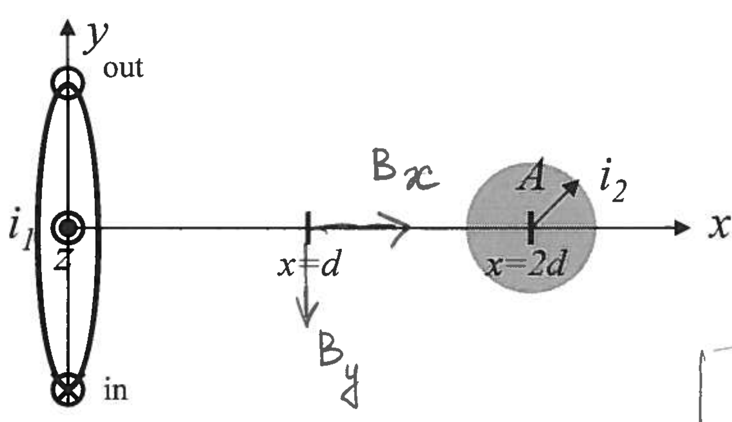
$$B = \frac{\mu_0 i_2 r}{2\pi A^2} \quad \text{CCW}$$

$r > A$

$$B\,2\pi r = \mu_0 i_2$$

$$B = \frac{\mu_0 i_2}{2\pi r} \quad \text{CCW}$$

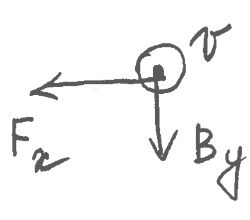
c) A **negatively** charged particle, $-q_0$ is at point $x = d$ with a velocity $\vec{v} = v_0\vec{i}_z$, where v_0 is a constant. Find the force acting on the particle at $x = d$.

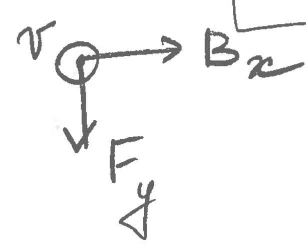


$$F_x = -|q_0|\, v_0 \frac{\mu_0 i_2}{2\pi d}$$

$$F_y = -|q_0|\, v_0 \frac{\mu_0 i_1 R^2}{2(R^2+d^2)^{3/2}}$$

$$\boxed{\vec{F} = F_x \vec{i}_x + F_y \vec{i}_y}$$

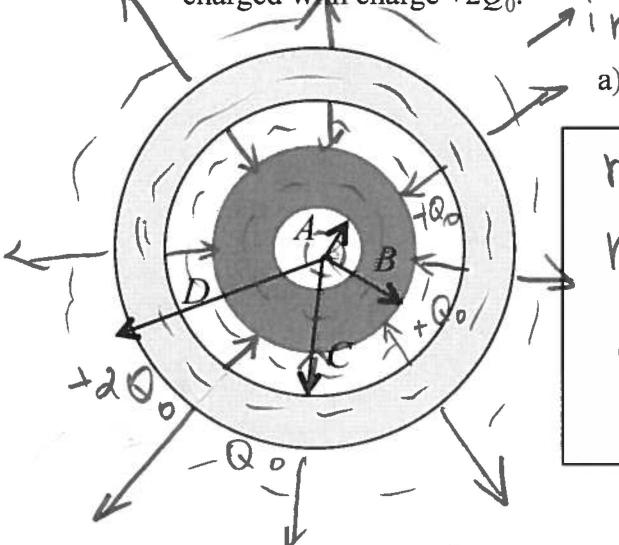




negative charge!

Problem 5: (20 points)

A spherical conducting shell, inner radius A and outer radius B, is charged with charge $-Q_0$. It is surrounded by a conducting spherical shell of inner radius C and outer radius D, which is charged with charge $+2Q_0$.



a) Find the charge per unit area on all four surfaces.

$r = A \quad \sigma = 0$

$r = B \quad \sigma = -\dfrac{Q_0}{4\pi B^2}$

$r = C \quad \sigma = \dfrac{Q_0}{4\pi C^2}$

$r = D \quad \sigma = \dfrac{Q_0}{4\pi D^2}$

b) Find the electric field at

i) $r < A$

$E = 0 \quad Q_{encl} = 0$

ii) $A < r < B$

$E = 0$ (conductor)

iii) $B < r < C$

$$E\,4\pi r^2 = -\frac{Q_0}{\varepsilon_0}\ ; \quad E_r = -\frac{Q_0}{4\pi\varepsilon_0 r^2}$$

iv) $C < r < D$

$E = 0$ (conductor)

v) $r > D$ $\quad E\,4\pi r^2 = \dfrac{2Q_0 - Q_0}{\varepsilon_0}\ ; \quad E_r = \dfrac{Q_0}{4\pi\varepsilon_0 r^2}$

c) Sketch the electric field lines.

d) Find the difference in electric potential between $r = 0$ and $r = \infty$, $V(\infty) - V(0)$.

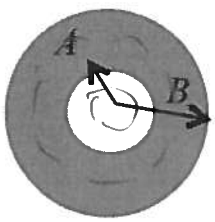
$$V(\infty) - V(0) = -\int_0^\infty \vec{E}\cdot d\vec{r} = -\left[\int_0^A 0 + \int_A^B 0 + \int_B^C \left(-\frac{Q_0}{4\pi\varepsilon_0 r^2}\right) dr\right.$$

$$\left. + \int_C^D 0 + \int_D^\infty \frac{Q_0}{4\pi\varepsilon_0 r^2}\, dr\right] = -\frac{Q_0}{4\pi\varepsilon_0}\left(\frac{1}{C} - \frac{1}{B}\right) - \frac{Q_0}{4\pi\varepsilon_0 D}$$

d) A nonconducting spherical shell, inner radius A and outer radius B, has a nonuniform volume charge density $\rho = cr$ where c is a known positive constant.

a) Find the electric field at

1) $r < A$



$E = 0 \qquad Q_{encl} = 0$

2) $A < r < B$

$$E 4\pi r^2 = \frac{1}{\varepsilon_0}\int_A^r c r \, 4\pi r^2 dr = \frac{1}{\varepsilon_0} 4\pi c \left.\frac{r^4}{4}\right|_A^r =$$

$$= \frac{1}{\varepsilon_0}\pi c \left(r^4 - A^4\right)$$

$$E = \frac{\pi c}{\varepsilon_0 4\pi r^2}\left(r^3 - A^3\right) = \frac{c}{4\varepsilon_0}\frac{r^4 - A^4}{r^2}$$ rad out

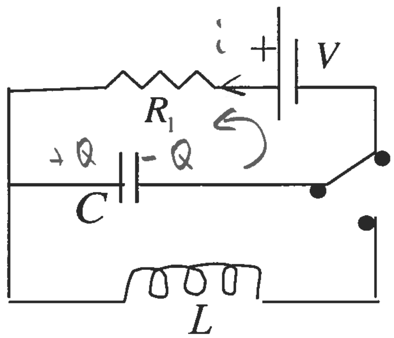
3) $r > B$

$$E 4\pi r^2 = \frac{1}{\varepsilon_0}\int_A^B c r \, 4\pi r^2 dr = \frac{1}{\varepsilon_0}\frac{4\pi c}{4}\left(B^4 - A^4\right)$$

$$E = \frac{c}{4\varepsilon_0}\frac{B^4 - A^4}{r^2}$$ rad out

Problem 6: (20 points)

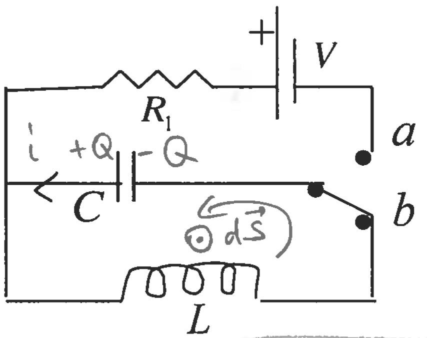
a) In a circuit below, R, C, L, and V are given. The switch is kept at position a for a long time. Find the current and the charge on the capacitor. *The problem will not be graded without a direction of current and charges on the capacitor indicated on the circuit.*



$i = 0 \qquad -V + \frac{Q}{C} = 0$

$Q = CV$

b) The switch is then thrown to position b. Starting from some famous law, derive the equation that describes charge Q on the capacitor as a function of time. $Q(t=0) = CV$



$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$

$\Phi = Li; \quad \frac{d\Phi}{dt} = L\frac{di}{dt}; \quad i = -\frac{dQ}{dt}$

$-\frac{Q}{C} = -L\frac{di}{dt}; \quad L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0$

$$\boxed{\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0}$$

b) Solve for the charge on the capacitor as a function of time.

$Q(t) = A\cos\omega t + B\sin\omega t$

$Q(t=0) = A = CV$

$i = -\frac{dQ}{dt} = -(-A\omega\sin\omega t + B\omega\cos\omega t)$

$i(t=0) = -B\omega = 0$

$Q(t) = CV\cos\omega t$

$\frac{d^2Q}{dt^2} = -CV\omega^2\cos\omega t$

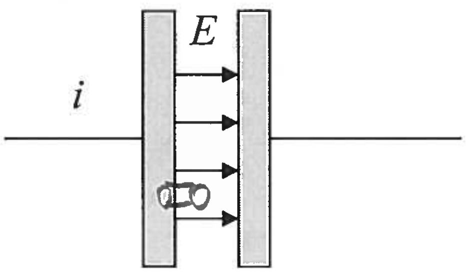
$-CV\omega^2\cos\omega t + \frac{1}{LC}CV\cos\omega t = 0$

$\omega^2 = \frac{1}{LC}$

$$\boxed{Q(t) = CV\cos\left(\frac{1}{\sqrt{LC}}t\right)}$$

Problem 7: (7 points)

a) Find the displacement current and the current i in the wire if the electric field between the parallel plates of the capacitor is $E=E_0\sin(\omega t)$. The area of the plates is A.



$$i_D = \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{s}$$

$$\Phi_E = \int \vec{E} \cdot d\vec{s} = E_0 \sin(\omega t) A$$

$$\frac{d\Phi_E}{dt} = E_0 \omega \cos(\omega t) A$$

$$\boxed{i_D = \varepsilon_0 E_0 \omega \cos(\omega t) A}$$

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E a = \frac{\sigma a}{\varepsilon_0} \; ; \quad E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A \varepsilon_0}$$

$$Q = E A \varepsilon_0 = \varepsilon_0 E_0 \sin(\omega t) A$$

$$\boxed{i = \frac{dQ}{dt} = \varepsilon_0 E_0 \omega \cos(\omega t) A}$$