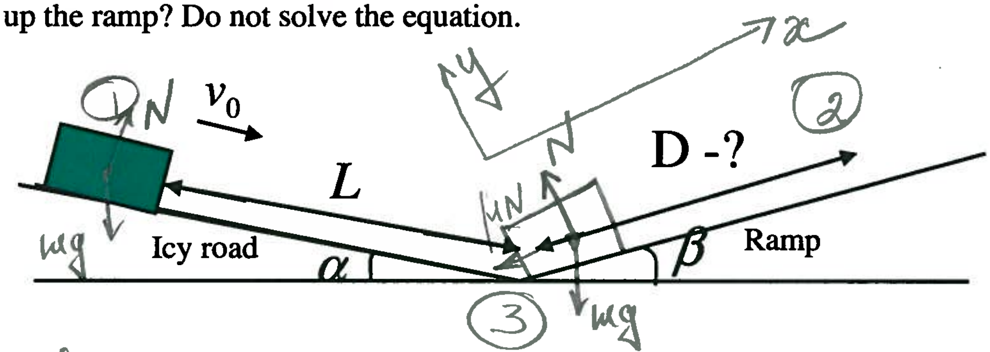
Problem 1:

A sled of mass m slides down an icy mountain road of constant downward slope angle α. At t=0 the sled starts moving downhill at speed v_0. After careening downhill a distance L with negligible friction, the sled runs onto a ramp of constant upward slope β.

a) If the ramp has a soft sand surface for which the coefficient of friction $\mu = \mu_0(1+x/c)$, where μ_0 and c are known constants, x is the distance along the ramp, what is the distance that the sled moves up the ramp? Do not solve the equation.



$$W^{fric} = \left[U_2 + \frac{m v_2^2}{2}\right] - \left[U_1 + \frac{m v_1^2}{2}\right]$$

$$N - mg\cos\beta = 0$$

$$N = mg\cos\beta$$

$$W^{fric} = -\int_0^D mg\cos\beta\,\mu_0\left(1+\frac{x}{c}\right)dx =$$

$$= -mg\cos\beta\,\mu_0\left(x + \frac{x^2}{2c}\right)\Big|_0^D =$$

$$= -mg\cos\beta\,\mu_0\left(D + \frac{D^2}{2c}\right)$$

$$U_2 = mgD\sin\beta; \quad U_1 = mgL\sin\alpha$$

$$\boxed{-g\cos\beta\,\mu_0\left(D + \frac{D^2}{2c}\right) = gD\sin\beta - gL\sin\alpha - \frac{v_0^2}{2}}$$

b) If the ramp has the coefficient of friction $\mu = \mu_0(1+t/c)$, where t is time, find the distance that the sled moves up the ramp.

$$\frac{m v_0^2}{2} + mgL\sin\alpha = \frac{m v_3^2}{2}$$

$$v_3 = \sqrt{v_0^2 + 2gL\sin\alpha}$$

$$F_x = m a_x$$

$$-mg\sin\beta - \mu_0\left(1+\frac{t}{c}\right)mg\cos\beta = m a_x$$

$$a_x = -g\sin\beta - \mu_0\left(1+\frac{t}{c}\right)g\cos\beta$$

$$v_x = \int a_x\,dt = -\int\left(g\sin\beta + \mu_0\left(1+\frac{t}{c}\right)g\cos\beta\right)dt =$$

$$= -g\sin\beta\, t - \mu_0 g\cos\beta\, t - \frac{\mu_0 g\cos\beta}{c}\frac{t^2}{2} + v_3$$

$$x(t) = \int v_x(t)\,dt = \int\left(-g\sin\beta\, t - \mu_0 g\cos\beta\, t - \frac{\mu_0 g\cos\beta}{c}\frac{t^2}{2} + v_3\right)dt = -g(\sin\beta + \mu_0\cos\beta)\frac{t^2}{2} - \frac{\mu_0 g\cos\beta}{6c}t^3 + v_3 t$$

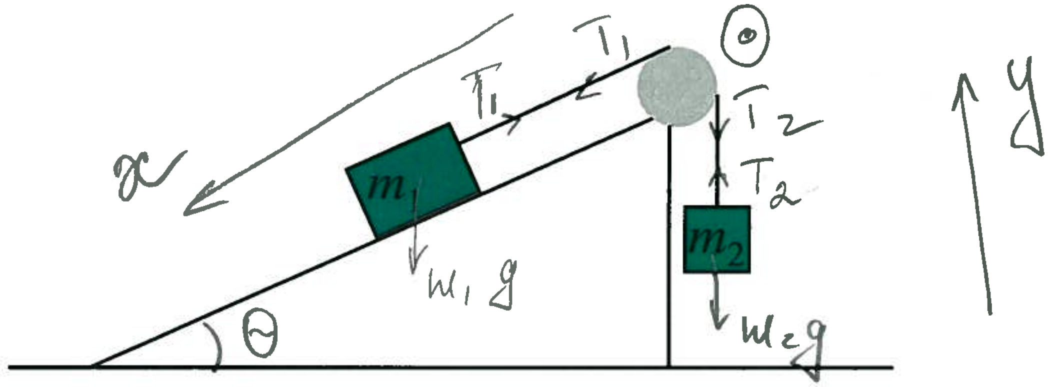
$$x(0) = 0$$

$$v_x(t^*) = 0 \Rightarrow t^*$$

$$\boxed{D = x(t^*) = -g(\sin\beta + \mu_0\cos\beta)\frac{t^{*2}}{2} - \frac{\mu_0 g\cos\beta}{6c}t^{*3} + v_3 t^*}$$

Problem 2:

Block of mass m_1 is placed on the frictionless inclined plane of angle θ. It is connected to the block of mass m_2 by unstretchable string of negligible mass. The string pulls without slipping. If the pulley of radius R has moment of inertia I with respect to the axis of rotation, find the acceleration of the block m_1.



$$\begin{cases} m_1 g \sin\theta - T_1 = m_1 a_{1x} \\ T_2 - m_2 g = m_2 a_{2y} \\ a_{1x} = a_{2y} = a \\ T_1 R - T_2 R = I\alpha \\ a = R\alpha \end{cases}$$

$$T_1 - T_2 = \frac{I\alpha}{R}$$

$$m_1 g \sin\theta - (T_1 - T_2) - m_2 g = (m_1 + m_2) a$$

$$m_1 g \sin\theta - \frac{Ia}{R^2} - m_2 g = (m_1 + m_2) a$$

$$\boxed{a = \frac{(m_1 \sin\theta - m_2) g}{m_1 + m_2 + \frac{I}{R^2}}}$$

Problem 3:

A block of mass *m* is connected to a very light spring with constant *k*. The spring is compressed by amount of *L*. The surface is frictionless.

a) How far will the block move if it was given initial velocity v_0?

$$\frac{m v_0^2}{2} + \frac{k L^2}{2} = \frac{k x_s^2}{2}$$

$$x_s = \sqrt{\frac{m v_0^2 + k L^2}{k}}$$

b) Show that the motion of the block is described by the equation of harmonic oscillator.

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$x(0) = A = -L$$

$$\frac{dx}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\frac{dx}{dt}(0) = B\omega = v_0$$

$$x(t) = -L\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

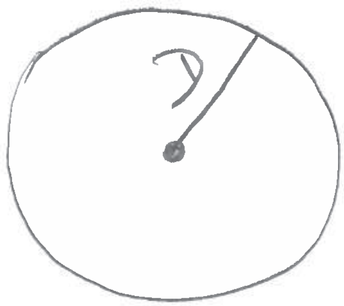
c) Find the period of oscillations.

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Problem 4:

A very small object, mass m, is attracted to the origin by a force of magnitude $C\frac{m}{r^4}$, where r is the distance from the origin and C is a known constant.

a) What initial velocity would the object have to be given in order for it to move in a circle if it was originally a distance D from the origin?



$$F_r = m a_r$$

$$-C\frac{m}{D^4} = -m\frac{v^2}{D}$$

$$\boxed{v = \sqrt{\frac{C}{D^3}}}$$

b) What is the object's angular momentum about the origin?

$$\vec{L} = \vec{r} \times \vec{p}$$

$L = mDv = mD\sqrt{\frac{C}{D^3}}$ ($\perp$ to the plane of the paper)

c) What would be the change in the potential energy of the object due to this force if it moved from the point $r=D$ and $\theta=0$ to the point $r=4D$ and $\theta=\pi$?

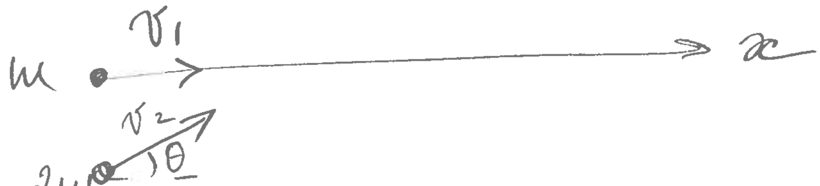
$$F_r = -\frac{dU}{dr};$$

$$U = -\int F_r\, dr = -\int -C\frac{m}{r^4}\, dr = -\frac{Cm}{3r^3} + C$$

$$U(D) - U(4D) = -\frac{Cm}{3D^3} + \frac{Cm}{3 \cdot 64 D^3}$$

Problem 5:

Two blocks are sliding on a frictionless table. One, with mass m, has a known velocity of magnitude v_1 in the $+x$ direction. The second, mass $2m$, has a velocity that is at the known angle θ with the x axis and has unknown magnitude, call it v_2. They collide and stick together and move away from the point of collision with a velocity which makes an angle A with the x axis. What was the original velocity of the second block and the magnitude of the final velocity?



$$P_x(\text{before}) = P_x(\text{after})$$
$$P_y(\text{before}) = P_y(\text{after})$$

$$\begin{cases} m v_1 + 2m v_2 \cos\theta = 3m u \cos A \\ 0 + 2m v_2 \sin\theta = 3m u \sin A \end{cases}$$

$$v_2 = \frac{3u \sin A}{2\sin\theta}$$

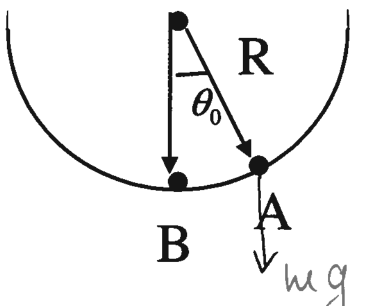
$$m v_1 + 2m \frac{3u\sin A}{2\sin\theta}\cos\theta = 3m u \cos A$$

$$u = \frac{v_1}{3(\cos A - \sin A \cot\theta)}$$

$$v_2 = \frac{v_1}{(\cos A - \sin A\cot\theta)}\,\frac{\sin A}{2\sin\theta}$$

Problem 6

A small rock with mass m is released from rest at point A (angle θ_0 from the vertical). The hemispherical bowl has radius R. Assume that the size of the rock is small compared to R, so that the rock can be treated as a particle, and assume that the rock slides rather than rolling. The surface of the bowl is frictionless. 1) Find the position of the rock as a function of time ($\theta(t)$). (Point A is very close to point B, you can use the small angle approximation $\sin\theta \sim \theta$).



$$\vec{\tau}_{ext} = I\alpha \quad (rhr)$$

$$-mgR\sin\theta \approx \theta = mR^2\alpha = mR^2\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{R}\theta = 0$$

$$\theta(t) = A\cos\omega t + B\sin\omega t \;; \quad \omega = \sqrt{\frac{g}{R}}$$

$$\theta(0) = A = \theta_0$$

$$\frac{d\theta}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\frac{d\theta}{dt}(t=0) = B\omega = 0 \Rightarrow B = 0$$

$$\boxed{\theta(t) = \theta_0\cos\sqrt{\frac{g}{R}}\,t}$$

2) How long will it take for the rock to return to the point B ~~at the bottom of the~~ bowl?

$$\frac{1}{4}\text{Period} = \frac{1}{4}\frac{2\pi}{\omega} = \boxed{\frac{\pi}{2}\sqrt{\frac{R}{g}}}$$

or

$$\cos\omega t = 0 \Rightarrow \omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega} = \frac{\pi}{2}\sqrt{\frac{R}{g}}$$