**Problem 1: (15 points)**

A model airplane with mass $m$ moves in the xy-plane such that its x- and y- coordinates vary in time according to $x(t) = \alpha - \beta t^3$ and $y(t) = \gamma t - \delta t^2$ where $\alpha, \beta, \gamma,$ and $\delta$ are known constants.

a) Calculate the x- and y-components of the net force on the plane as functions of time.

$$v_x(t) = \frac{dx}{dt} = -3\beta t^2$$

$$v_y(t) = \frac{dy}{dt} = \gamma - 2\delta t$$

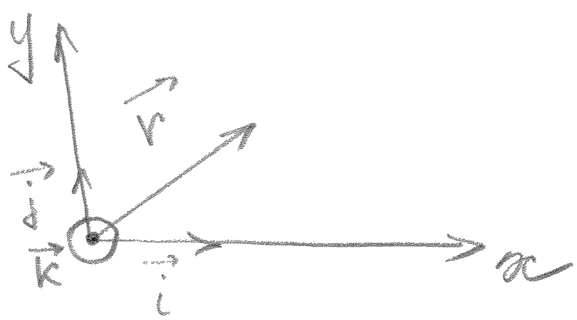
$$a_x(t) = \frac{dv_x(t)}{dt} = -6\beta t$$

$$a_y(t) = \frac{dv_y(t)}{dt} = -2\delta$$

$$F_x = -m \cdot 6\beta t$$

$$F_y = -m \cdot 2\delta$$

b) Find the angular momentum of the plane about the origin as a function of time.




$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\vec{p} = m(v_x\vec{i} + v_y\vec{j})$$

$$\vec{L} = \vec{r} \times \vec{p} = (x\vec{i} + y\vec{j}) \times m(v_x\vec{i} + v_y\vec{j}) =$$

$$= m(xv_y - yv_x)\vec{k} = m\left((\alpha - \beta t^3)\cdot(\gamma - 2\delta t) + \right.$$

$$\left. + (\gamma t - \delta t^2)\, 3\beta t^2\right)\vec{k}$$

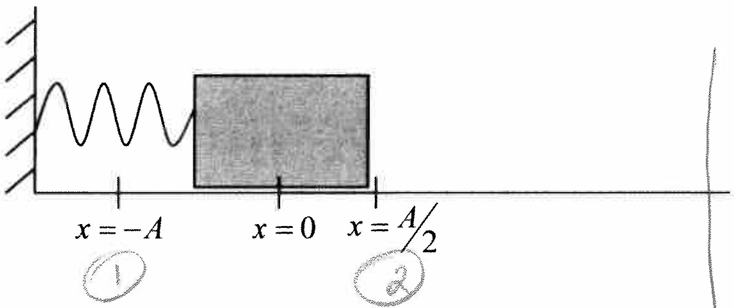
## Problem 2: (20 points)

A certain spring is found not to obey Hooke's law; it exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ if it is stretched or compressed, where $\alpha$ and $\beta$ are known constants. The mass of the spring is negligible.

a) Calculate the potential energy function $U(x)$ for the spring.

$$U(x) = -\int F_x\,dx = -\int(-\alpha x - \beta x^2)\,dx = \frac{\alpha x^2}{2} + \frac{\beta x^3}{3} + C$$

b) An object with mass m on a frictionless, horizontal surface attached to this spring is pulled a a distance A to the left (in the $-x$ direction) and then released. If the surface is frictionless, find the velocity of the object when it is A/2 to the right of the equilibrium position.




$$U_1 + KE_1 = U_2 + KE_2$$

$$U_1 = \frac{\alpha A^2}{2} - \frac{\beta A^3}{3}$$

$$KE_1 = 0 \qquad KE_2 = \frac{mv^2}{2}$$

$$U_2 = \frac{\alpha\left(\frac{A}{2}\right)^2}{2} + \frac{\beta\left(\frac{A}{2}\right)^3}{3}$$

$$\frac{\alpha A^2}{2} - \frac{\beta A^3}{3} = \frac{\alpha\left(\frac{A}{2}\right)^2}{2} + \frac{\beta\left(\frac{A}{2}\right)^3}{3} + \frac{mv^2}{2}$$

$$v = \sqrt{\frac{2}{m}\left(\frac{\alpha A^2}{2} - \frac{\beta A^3}{3} - \frac{\alpha A^2}{8} - \frac{\beta A^3}{24}\right)} = \sqrt{\frac{1}{m}\left(\frac{3}{4}\alpha A^2 - \frac{3}{4}\beta A^3\right)}$$

c) How the answer in b) will change is there is a coefficient of friction between the block and the surface, μ. (Assume that the block will reach this point)

$$W^{frict} = [U_2 + KE_2] - [U_1 + KE_1]; \quad F_{fr} = \mu N = \mu mg$$

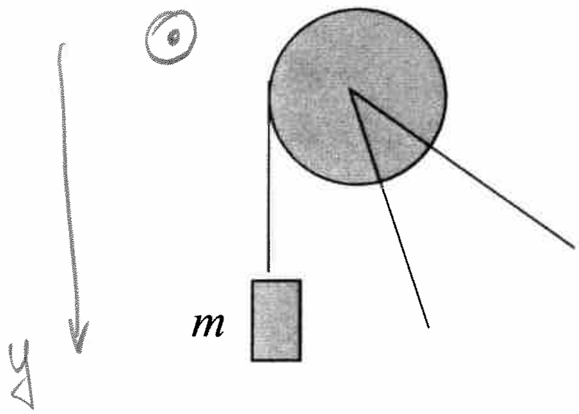
$$-\int_{-A}^{\frac{A}{2}} \mu mg\,dx = [U_2 + KE_2] - [U_1 + KE_1]$$

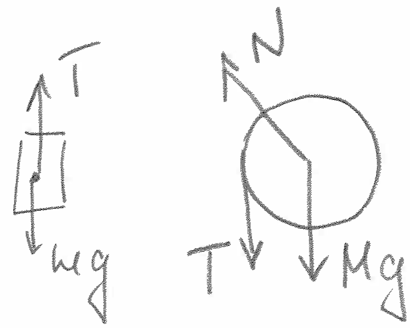
$$-\mu mg\left(\frac{A}{2} + A\right) = [U_2 + KE_2] - [U_1 + KE_1]$$

$$-\frac{3}{2}\mu mg A = [U_2 + KE_2] - [U_1 + KE_1]$$

**Problem 3: (20 points)**

A mass *m* is hung from a massless, unstretchable string which is wrapped around a disk with radius *R* and mass *M*. The string pulls without slipping.




$F_y = m a_y$ ; $\vec{\tau}_{ext} = I\alpha$ (rhr

a) If the axle is frictionless, find the tension in the string.

$$\begin{cases} mg - T = m a_y \\ TR = \dfrac{MR^2}{2}\alpha \\ a_y = R\alpha \end{cases}$$

$$a_y = R\frac{2T}{MR}$$

$$T = mg - mR\frac{2T}{MR}; \quad \boxed{T = \frac{mg}{1 + \frac{2m}{M}}}$$

b) Find the acceleration of the mass.

$$a_y = R\alpha = \frac{2mg}{M\left(1 + \frac{2m}{M}\right)}$$

c) What will be the angular velocity of the disk as a function of time, assuming that at t=0 the system was at rest? $(\omega_0 = 0)$

$$\omega = \alpha t = \frac{2mgt}{MR\left(1 + \frac{2m}{M}\right)}$$

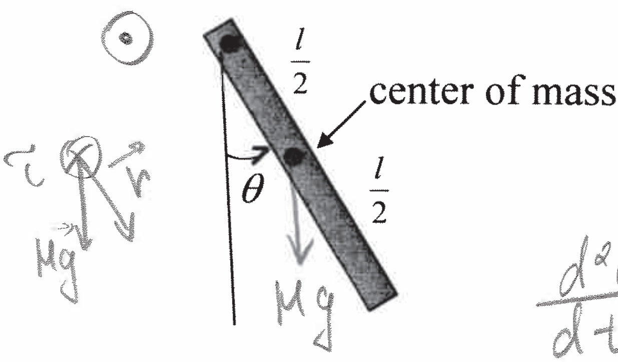
d) Suppose there is a friction force at the axle which exerts a torque, opposing the rotation. This torque is proportional to the angular velocity, with magnitude $\beta\omega$ where $\beta$ is a known constant. The pulley will then reach a certain constant angular velocity. What is this constant value of $\omega$?

$$\begin{cases} TR - \beta\omega - I\alpha = 0 \quad (\omega = \text{const}) \\ T = mg \quad (a = R\alpha = 0) \end{cases}$$

$$\boxed{\omega = \frac{mgR}{\beta}}$$

## Problem 4: (15 points)

A "physical pendulum" is any real object, free to rotate about some horizontal axis.
a) Show that a thin rod, length $l$, pivoted about one end with moment of inertia $I$ about this end, will undergo simple harmonic motion if displaced a small angle from equilibrium (You can use $\sin\theta \approx \theta$ ).




$$\vec{\tau}_{ext} = I\alpha \ (\text{rhr})$$

$$-Mg\frac{l}{2}\sin\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{Mgl}{2I}}_{=\omega^2}\theta = 0$$

$$\theta(t) = \theta_0 \cos\omega t$$

$$\frac{d\theta}{dt} = -\theta_0\omega\sin\omega t$$

$$\frac{d^2\theta}{dt^2} = -\theta_0\omega^2\cos\omega t$$

$$-\theta_0\omega^2\cos\omega t + \frac{Mgl}{2I}\theta_0\cos\omega t = 0$$

$$\omega^2 = \frac{Mgl}{2I}$$

b) What is the frequency of oscillations?

$$\omega = \sqrt{\frac{Mgl}{2I}} \ ; \quad f = \frac{\omega}{2\pi}$$

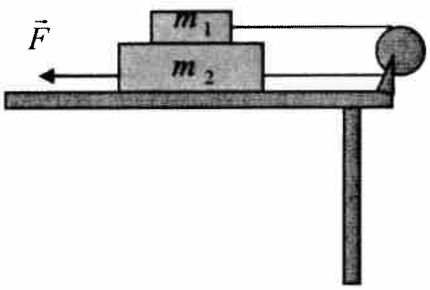
c) If the rod starts at a position $\theta_0$ from rest, how long will it take for the rod to return to its equilibrium?

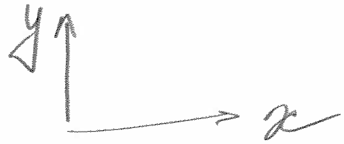
$$\cos\omega t = 0 \Rightarrow \omega t^* = \frac{\pi}{2}$$

$$t^* = \frac{\pi}{2\omega} = \frac{\pi\sqrt{2I}}{2\sqrt{Mgl}}$$

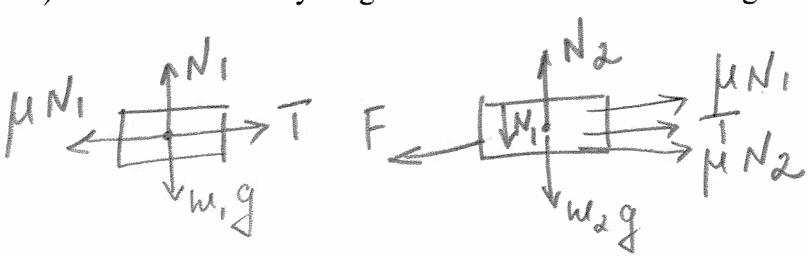
## Problem 5: (20 points)

Two blocks with masses $m_1$ and $m_2$ are connected by a light, unstretchable cord passing around a fixed, **frictionless** pulley. The coefficient of friction between two blocks and the block of mass $m_2$ and the surface is $\mu$.




a) Draw the free body diagram for both blocks assuming that block $m_2$ is moving to the left.




b) Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block $m_2$ to the left at a **constant velocity**.

$$F_x = m a_x \qquad F_y = m a_y$$

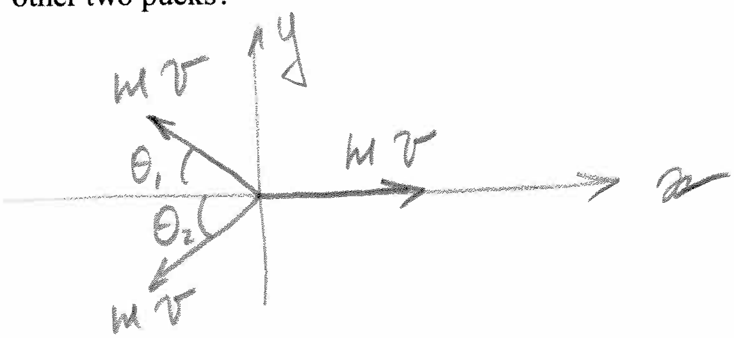
$$\begin{cases} T - \mu N_1 = m_1 a_x = 0 \quad (v = \text{Const}) \\ N_1 = m_1 g \\ T + \mu N_1 + \mu N_2 - F = 0 \quad (v = \text{Const}) \\ N_2 = N_1 + m_2 g = (m_1 + m_2) g \end{cases}$$

$$T = \mu m_1 g$$

$$F = \mu m_1 g + \mu m_1 g + \mu (m_1 + m_2) g = \mu g (3 m_1 + m_2)$$

**Problem 6: (15 points)**

Three identical pucks (each has the same mass) on a horizontal air table (frictionless) have repelling magnets. They are held together and then released simultaneously. Each has the same magnitude of velocity at any instant. One puck moves due east. What is the direction of the velocity of each of the other two pucks?




$$p_x(\text{before}) = p_x(\text{after})$$

$$p_y(\text{before}) = p_y(\text{after})$$

$$0 = mv\cos\theta_1 + mv\cos\theta_2 - mv$$

$$0 = mv\sin\theta_1 - mv\sin\theta_2 \Rightarrow \sin\theta_1 = \sin\theta_2$$

$$\theta_1 = \theta_2$$

$$mv\cos\theta_1 + mv\cos\theta_1 = mv$$

$$2\cos\theta_1 = 1$$

$$\cos\theta_1 = \frac{1}{2}$$

$$\theta_1 = 60^\circ$$

$$\theta_2 = 60^\circ$$