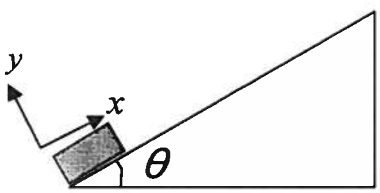
Problem 1: (20 points)

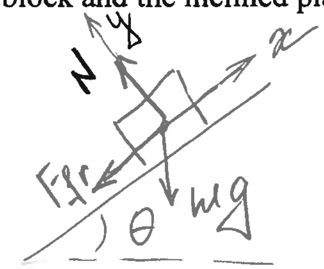
A block of mass m is given an initial velocity v_0 up an inclined plane with angle θ. In the coordinate system given, the coefficient of friction between the block and the plane is

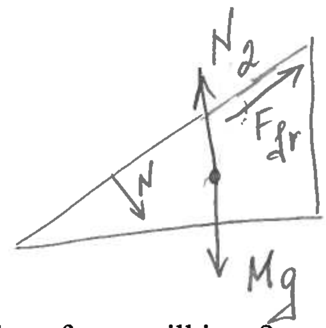
$$\mu = \mu_0\left(1+\frac{x}{c}\right)$$

where μ_0 and c are known constants.

a) Draw the Free Body Diagram for the block and the inclined plane.







b) Assuming the block stops before it reaches the top of the plane, how far up will it go?

$$W^{net} = \frac{m v_f^2}{2} - \frac{m v_i^2}{2} \quad ; \quad N = mg\cos\theta$$

$$W^{mg} = -mgh \quad \text{or} \quad W^{mg} = \int_0^{\frac{h}{\sin\theta}} -mg\sin\theta\, dx = -mgh$$

$$W^{N} = 0 \quad W^{fr} = -\int_0^{\frac{h}{\sin\theta}} \mu N\, dx = -\int_0^{\frac{h}{\sin\theta}} \mu_0\left(1+\frac{x}{c}\right) mg\cos\theta\, dx =$$

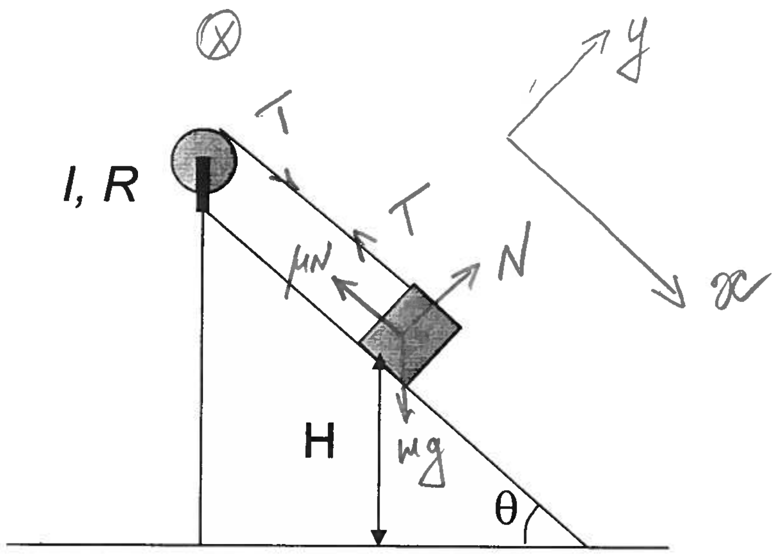
$$= -\mu_0\left(x + \frac{x^2}{2c}\right) mg\cos\theta \Big|_0^{\frac{h}{\sin\theta}} =$$

$$= -\mu_0\left(\frac{h}{\sin\theta} + \frac{h^2}{2c\sin^2\theta}\right) mg\cos\theta$$

$$\boxed{-\mu_0\left(\frac{h}{\sin\theta} + \frac{h^2}{2c\sin^2\theta}\right) mg\cos\theta - mgh = -\frac{m v_0^2}{2}}$$

Problem 2: (20 points)

A massless cord connected at one end to a block of mass m which can slide on an inclined plane of angle θ has its other end wrapped around a pulley of radius R and moment of inertia I around its axis, as shown in the figure. At $t = 0$ the block starts sliding down from initial height H above the bottom of the plane. Assume that the pulley rotates without friction and cord unwraps without slipping. The coefficient of friction between the block and the surface is μ.



a) Find the acceleration of the block.

$$\begin{cases} mg\sin\theta - T - \mu N = ma_x \\ N = mg\cos\theta \\ TR = I\alpha \quad \Rightarrow \quad T = \frac{I}{R}\alpha = \frac{I}{R^2}a_x \\ a_x = R\alpha \quad \Rightarrow \quad \alpha = \frac{a_x}{R} \end{cases}$$

$$mg\sin\theta - \frac{I}{R^2}a_x - \mu mg\cos\theta = ma_x$$

$$a_x = \frac{mg\sin\theta - \mu mg\cos\theta}{m + \frac{I}{R^2}}$$

b) Find the angular velocity of the pulley at the time when the block reaches the bottom of the wedge.

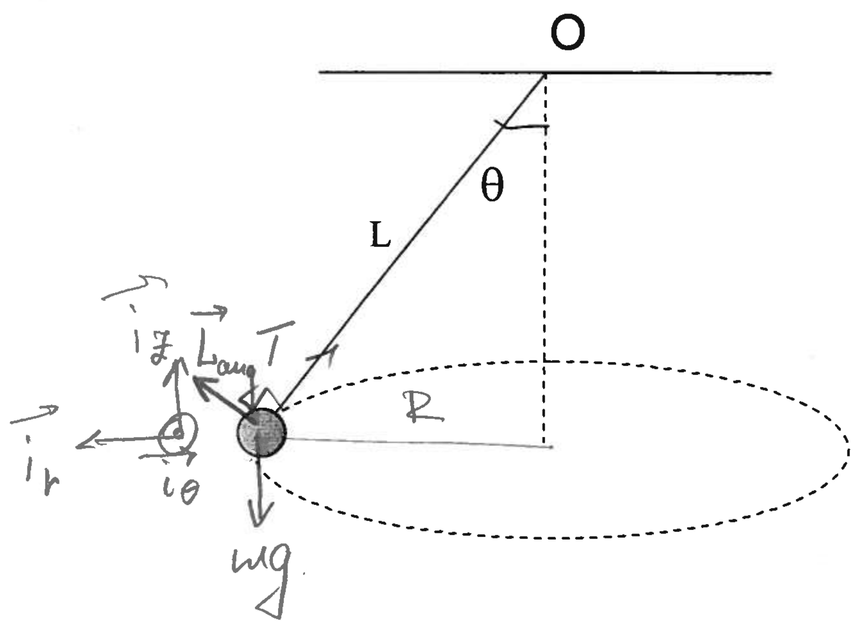
$$\omega = \alpha t$$

$$\frac{H}{\sin\theta} = \frac{a_x t^2}{2} \Rightarrow t = \sqrt{\frac{2H}{a_x\sin\theta}}\ ; \quad \omega = \frac{a_x}{R}\sqrt{\frac{2H}{a_x\sin\theta}}$$

Problem 3: (15 points)

A ball of mass *m* hanging on a massless string of length *L* rotates in a horizontal plane, so that the string makes an angle θ with vertical direction.

a) Draw a Free Body Diagram for the ball.



b) Find the angular velocity of rotation.

$$F_r = m a_r,$$

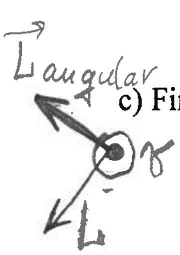
$$F_z = m a_z$$

$$-T \sin\theta = -m L \sin\theta\, \omega^2$$

$$T\cos\theta = mg \Rightarrow T = \frac{mg}{\cos\theta}$$

$$\cancel{m}g \tan\theta = \cancel{m} L \sin\theta\, \omega^2$$

$$\boxed{\omega = \sqrt{\frac{g}{L\cos\theta}}}$$



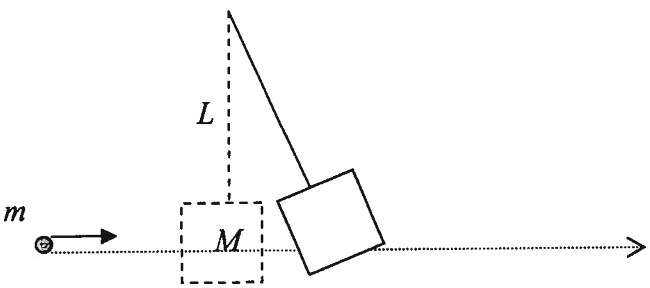
c) Find the angular momentum of the ball with respect to point O.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{L} \times \vec{p} = L \cdot m \overbrace{v_\theta}^{(rhr)} = L m \underbrace{L \sin\theta}_{R}\, \omega\, (rhr) =$$

$$= m L^2 \sin\theta\, \omega\ (rhr) - \text{see the figure)}$$

Problem 4 (20 points).

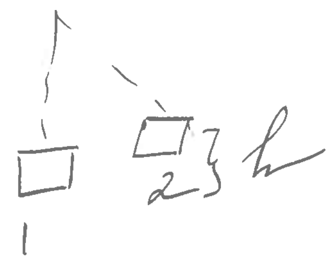
A bullet of mass m is shot *through* a wood block of mass M suspended on a massless unstretchable string of length L. The speed of the bullet is reduced by V_0 as a result of the impact. Assume the collision of the bullet and block takes place so quickly that the string remains vertical during the collision. Find h, the height at which the block stops.



$P_x(\text{before}) = P_x(\text{after})$

$m v_1 = m u_1 + M u_2$

$u_2 = \dfrac{m v_1 - m u_1}{M} = \dfrac{m V_0}{M}$



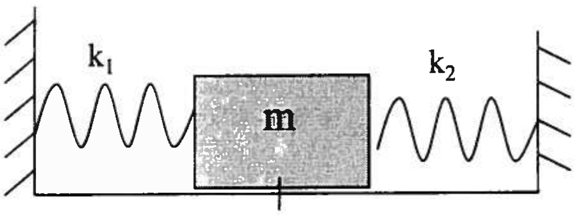
$U_1 + KE_1 = U_2 + KE_2$

$\dfrac{\cancel{M}}{2}\left(\dfrac{m V_0}{M}\right)^2 = \cancel{M} g h$

$$\boxed{h = \frac{1}{2g}\left(\frac{m V_0}{M}\right)^2}$$

Problem 5 (15 points).

Two springs are attached to a block of mass m on a frictionless table. Both springs are unstreched as shown. The block is pushed a distance x_0 to the right and released from rest.



a) Find the position of the block at any time.

$$-k_1 x - k_2 x = m \frac{d^2x}{dt^2}$$

$$m \frac{d^2x}{dt^2} + (k_1+k_2)x = 0$$

$$\frac{d^2x}{dt^2} + \frac{k_1+k_2}{m} x = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x(t=0) = A = x_0$$

$$v(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$v(t=0) = B\omega = 0 \Rightarrow B = 0$$

$$x(t) = x_0\cos\omega t \quad ; \quad -\cancel{x_0}\omega^2 \cancel{\cos\omega t} + \frac{k_1+k_2}{m}\cancel{x_0}\cancel{\cos\omega t} = 0$$

$$\omega = \sqrt{\frac{k_1+k_2}{m}}$$

b) What is the period of oscillations?

$$T = \frac{2\pi}{\omega}$$

Problem 6: (15 points)

a) (10 points) An object with mass m initially at rest is acted on by a single force $\vec{F} = k_1 t\vec{i} + k_2\vec{j}$, where k_1 and k_2 are known constants. Calculate the position of the object at any time.

$$F_x = k_1 t = m a_x$$

$$F_y = k_2 = m a_y$$

$$a_x = \frac{k_1}{m} t; \quad v_x = \int a_x \, dt = \frac{k_1}{m}\frac{t^2}{2} + c^{=0}$$

$$a_y = \frac{k_2}{m}; \quad v_y = \int a_y \, dt = \frac{k_2}{m} t + c^{=0}$$

$$x(t) = \int v_x(t)\, dt = \frac{k_1}{m}\frac{t^3}{6}$$

$$y(t) = \int v_y(t)\, dt = \frac{k_2 t^2}{2m}$$

b) (5 points) If, instead, the force acting on an object with mass m initially at a point L at rest is $\vec{F} = -k_1 x\vec{i}$, find the position of the object at any time.

$$-k_1 x = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + k_1 x = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x(t=0) = A = x_0$$

$$x(t) = x_0 \cos\omega t$$

$$\omega^2 = \frac{k_1}{m}$$