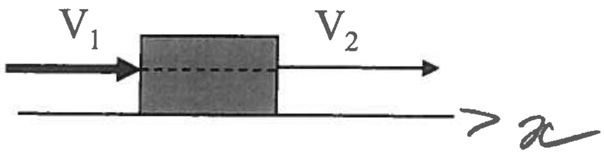
Problem 1: (20 points)

A bullet of mass m moving with a velocity V_1 hits a wooden block of mass M through its center and continues with velocity V_2 in the same direction.



a) Find the velocity of the box after the collision.

$$P_x(\text{before}) = P_x(\text{after})$$

$$mV_1 = mV_2 + MU$$

$$\boxed{U = \frac{m(V_1 - V_2)}{M}}$$

b) Find the kinetic energy that was lost in the collision.

$$KE_1 = \frac{mV_1^2}{2}$$

$$KE_2 = \frac{MU^2}{2} + \frac{mV_2^2}{2}$$

$$\boxed{KE_2 - KE_1 = \frac{MU^2}{2} + \frac{mV_2^2}{2} - \frac{mV_1^2}{2}}$$

c) After the collision the block continues moving for some time and then stops due to friction. Find the distance traveled by the block if the coefficient of friction is μ.

$$W_{net} = KE_2 - KE_1$$

$$W = \int_{x_1}^{x_2} F_x\,dx = \int_0^d -\mu N\,dx = \int_0^d -\mu Mg\,dx =$$

$$= -\mu Mgd$$

$$-\mu Mgd = 0 - \frac{MU^2}{2}\ ; \quad \boxed{d = \frac{U^2}{2\mu g}} = \frac{m^2(V_1 - V_2)^2}{2\mu g M^2}$$

Problem 2: (20 points)

This is a one-dimensional problem. A particle with mass m is acted on by a single force equal to

$$F_x(x) = -ax + b$$

where a and b are known positive constants. The particle is initially placed at $x = x_0$ and is given an initial velocity $V_x = V_0$, where x_0 and V_0 are known positive constants.

a) Verify that the force is conservative by calculating the potential energy function.

$F_x = -\frac{dU}{dx}$; $U = -\int(-ax + b)\,dx =$

$= \frac{ax^2}{2} - bx + \text{Const}$

b) Find the coordinates of the turning points where the direction of motion of a particle reverses. **Write down the equation but do not solve it!**

$KE_1 + U_1 = KE_2 + U_2$

turning points $v = 0$, $KE_2 = 0$

$$\boxed{\frac{ax^2}{2} - bx = \frac{ax_0^2}{2} - bx_0 + \frac{mv_0^2}{2}}$$

c) Find the point where the kinetic energy of the particle will have its maximum value.

KE is max when PE is min

$\frac{dU}{dx} = ax - b = 0 \Rightarrow \boxed{x = \frac{b}{a}}$

$\frac{d^2U}{dx^2} = a > 0$ min

d) Find the work done by this force as the particle moves from x_0 to $2x_0$.

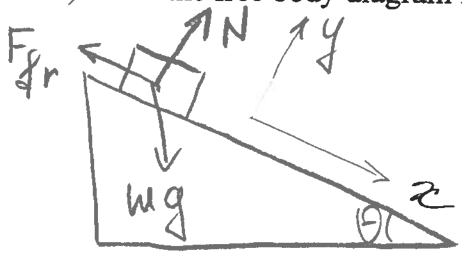
$W_{x_0 \to 2x_0} = -[U_2 - U_1] = -\left[\frac{a(2x_0)^2}{2} - 2bx_0 - \frac{ax_0^2}{2} + bx_0\right]$

$$= \boxed{-\frac{3}{2}ax_0^2 + bx_0}$$

Problem 3: (20 points)

A cylinder and a block are placed on the top of an inclined plane of height H and inclination angle θ. They start moving down simultaneously from rest. The coefficient of friction between the block and the incline is μ. A cylinder rolls without slipping. (Reminder: a cylinder of mass M and radius R has a moment of inertia $I = (1/2)\,MR^2$ around the central axis.)

a) Draw the free body diagram for the block.



b) Find the acceleration of the block.

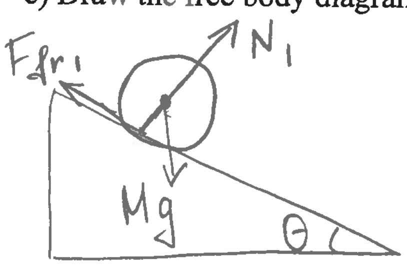
$$F_x = ma_x;\ F_y = ma_y$$

$$\begin{cases} mg\sin\theta - \mu N = ma_x \\ N - mg\cos\theta = 0 \end{cases}$$

$$mg\sin\theta - \mu\, mg\cos\theta = ma_x$$

$$\boxed{a_x^{block} = g(\sin\theta - \mu\cos\theta)}$$

c) Draw the free body diagram for the cylinder.



$$F_x = ma_x \qquad a_{cm} = R\alpha$$

$$F_y = ma_y$$

$$\tau_{ext} = I\alpha$$

d) Find the force of friction between the cylinder and the incline.

$$\begin{cases} Mg\sin\theta - F_{fr_1} = Ma_x \\ N = Mg\cos\theta \\ F_{fr}\cdot R = \dfrac{MR^2}{2}\alpha \\ a_{cm} = R\alpha \end{cases}$$

$$F_{fr} = \frac{MR}{2}\alpha = \frac{MR}{2}\frac{a_{cm}}{R} = \frac{M}{2M}\left(Mg\sin\theta - F_{fr}\right)$$

$$F_{fr} = \frac{Mg\sin\theta}{2} - \frac{F_{fr}}{2};\quad \boxed{F_{fr} = \frac{Mg\sin\theta}{3}}$$

e) Find the acceleration of the cylinder.

$$Mg\sin\theta - \frac{Mg\sin\theta}{3} = Ma_x$$

$$\boxed{a_x^{cyl} = \frac{2}{3} g \sin\theta}$$

f) Find the value of an inclination angle θ for which the two bodies reach the bottom of the incline at the same time.

$$x^{block} = \frac{1}{2} a_x^{block} t^2$$

$$x^{cyl} = \frac{1}{2} a_x^{cyl} t^2$$

$$x^{block} = x^{cyl} \quad \text{or} \quad a_x^{block} = a_x^{cyl}$$

$$\cancel{g}(\sin\theta - \mu\cos\theta) = \frac{2}{3}\cancel{g}\sin\theta$$

$$\frac{3}{2}\tan\theta - \tan\theta = \frac{3}{2}\mu$$

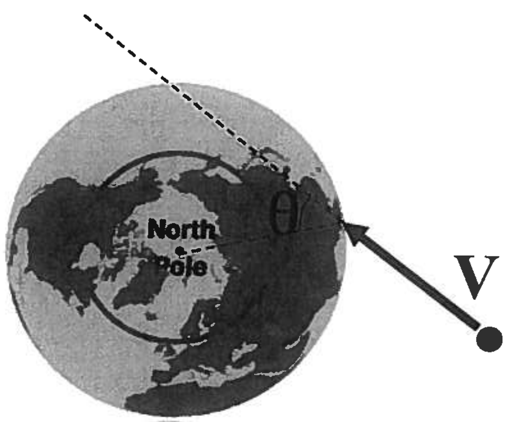
$$\frac{1}{2}\tan\theta = \frac{3}{2}\mu$$

$$\boxed{\tan\theta = 3\mu}$$

Problem 4: (15 points)

An asteroid of mass m strikes the Earth at the equator with a velocity V, as shown below (θ is a given constant). Assume that the asteroid remains stuck where it hit the ground. Assume that the Earth is a sphere of radius R, mass M, and moment of inertia I. It was rotating with angular velocity ω_0 before the collision. Find the angular velocity of the Earth after the collision.

View from the North pole. Note that the Earth rotates counter-clockwise as viewed from the North pole.



$$\vec{L} = \text{Const}$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$I\omega_0 + m v R \sin\theta = I\omega + mR^2\omega$$

$$\boxed{\omega = \frac{I\omega_0 + m v R \sin\theta}{I + mR^2}}$$

Problem 5: (15 points)

An object of mass m circles the Earth and is attracted to it with a force of magnitude given by

$$|\vec{F}| = G\frac{M_E m}{r^2}$$

where G and M_E are known constants.

a) If the object has a velocity of magnitude V_0, find the radius of the orbit.

$$F_r = m a_r$$

$$-G\frac{M_E m}{r^2} = -m\frac{v_0^2}{r}$$

$$\boxed{r = \frac{GM_E}{v_0^2} \equiv R_1}$$

b) Denote the radius you found in part a) as R_1. Find the work done by force F if you move the object from the orbit of radius R_1 to the orbit of radius $3R_1$.

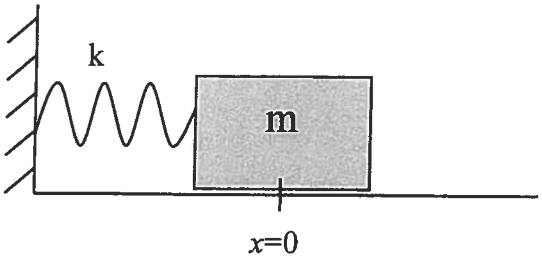
$$W = \int_{R_1}^{3R_1} -\frac{GM_E m}{r^2}\,dr = \left.\frac{GM_E m}{r}\right|_{R_1}^{3R_1} =$$

$$= GM_E m\left(\frac{1}{3R_1} - \frac{1}{R_1}\right) = -\frac{2}{3}\frac{GM_E m}{R_1} =$$

$$= \boxed{-\frac{2}{3} m v_0^2}$$

Problem 6 (15 points).

A spring with constant k is attached to a block of mass m on a frictionless table. The block is pushed a distance x_0 to the right and released from rest.



a) Find the position of the block at any time.

$$-kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x(t=0) = A = x_0$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$v(t=0) = B\omega = 0$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos\omega t - \overset{=0}{B}\omega\sin\omega t$$

$$-\cancel{A}\omega^2\cancel{\cos\omega t} + \frac{k}{m}\cancel{A}\cancel{\cos\omega t} = 0$$

$$\boxed{\omega^2 = \frac{k}{m}} \quad \boxed{x(t) = x_0\cos\sqrt{\frac{k}{m}}\,t}$$

b) How long will it take for the block to return from distance x_0 to its equilibrium position?

equilibrium: $x = 0 \Rightarrow \cos\omega t = 0$

$$\omega t = \frac{\pi}{2}; \quad t^* = \frac{\pi}{2\omega} = \boxed{\frac{\pi\sqrt{m}}{2\sqrt{k}}}$$

c) How fast will the block be moving at x=0?

$$x = 0 \quad t = \frac{\pi}{2\omega};$$

$$v\left(\frac{\pi}{2\omega}\right) = -x_0\omega = \boxed{-x_0\sqrt{\frac{k}{m}}}$$

Or:

$$\frac{kx_0^2}{2} = \frac{mv^2}{2}; \quad v = -x_0\sqrt{\frac{k}{m}}$$

d) What is the period of oscillations?

$$T = \frac{2\pi}{\omega} = \boxed{\frac{2\pi\sqrt{m}}{\sqrt{k}}}$$