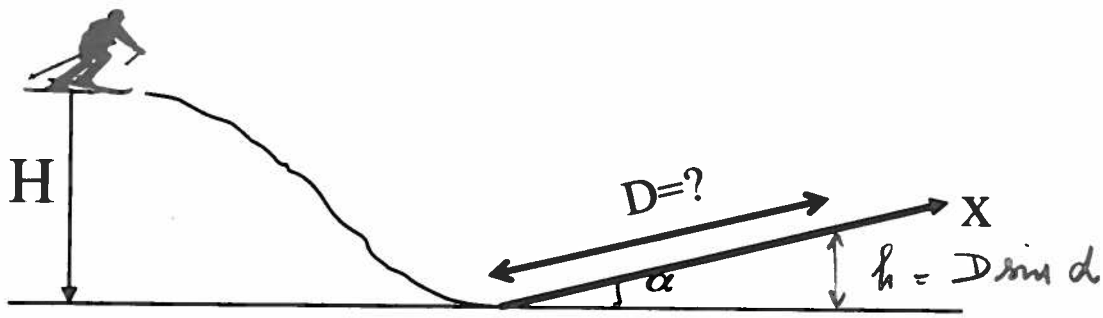
Problem 1: (20 points)

A skier with mass m starts from rest at the top of the frictionless ski slope H meters high. He immediately loses control over his skis and goes straight downhill. Fortunately, at the bottom of the slope he enters an upward ramp of constant slope angle α. The ramp has a soft snow surface with a coefficient of friction μ.

a) What is the distance, D, that the skier moves up the ramp before coming to a halt?



$$W^{non-cons} = [U_2 + KE_2] - [U_1 + KE_1]$$

$$\int_0^D (-\mu N)\,dx = mgD\sin\alpha - mgH$$

$$U_1 = mgH,\ KE_1 = 0,\ U_2 = mgD\sin\alpha,\ KE_2 = 0$$

$$N - mg\cos\alpha = 0$$

$$-\mu mg\cos\alpha D = mgD\sin\alpha - mgH$$

$$\boxed{D = \frac{H}{\sin\alpha + \mu\cos\alpha}}$$

b) Find this distance D if the coefficient of friction is given by $\mu = \mu_0 + cx^2$ where c is a known constant and x is counted from the bottom of the ramp along its surface. Stop when you have one equation for one unknown. Don't solve it.

$$W_{fric} = \int_0^D -(\mu_0 + cx^2)\, mg\cos\alpha\, dx =$$

$$= -\mu_0 mg\cos\alpha D - mg\cos\alpha\, c\,\frac{D^3}{3}$$

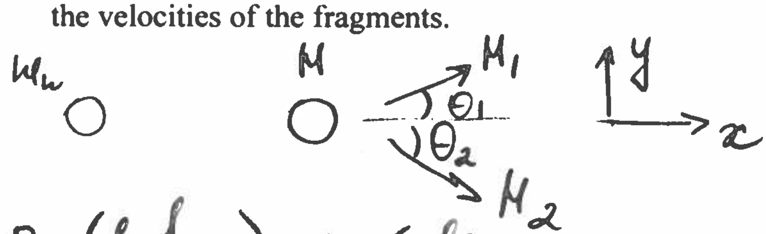
$$-\mu_0 mg\cos\alpha D - mg\cos\alpha\, c\,\frac{D^3}{3} = mgD\sin\alpha - mgH$$

$$\boxed{-\mu_0\cos\alpha D - \frac{cD^3}{3}\cos\alpha = D\sin\alpha - H}$$

Problem 2: (15 points)

A neutron of mass m_n moving with kinetic energy K_n hits a plutonium nucleus of mass M that was initially at rest. As a result of collision, the plutonium nucleus absorbs the neutron and decays into two fragments of masses M_1 and M_2 flying at the angles θ_1 and θ_2 with respect to the direction of motion of the incoming neutron.

a) In the box below write the system of equations that can be solved to find the magnitudes of the velocities of the fragments.



P_x (before) = P_x (after)

P_y (before) = P_y (after)

$$\boxed{\begin{aligned} \frac{m_n v_n^2}{2} &= K_n \\ m_n v_n &= M_1 u_1 \cos\theta_1 + M_2 u_2 \cos\theta_2 \\ 0 &= M_1 u_1 \sin\theta_1 - M_2 u_2 \sin\theta_2 \end{aligned}}$$

b) Find the magnitudes of the velocities of the fragments.

$$v_n = \sqrt{\frac{2K_n}{m}}$$

$$\boxed{u_1 = \frac{M_2 u_2 \sin\theta_2}{M_1 \sin\theta_1}}$$

$$m_n v_n = M_1 \frac{M_2 u_2 \sin\theta_2}{M_1 \sin\theta_1} \cos\theta_1 + M_2 u_2 \cos\theta_2$$

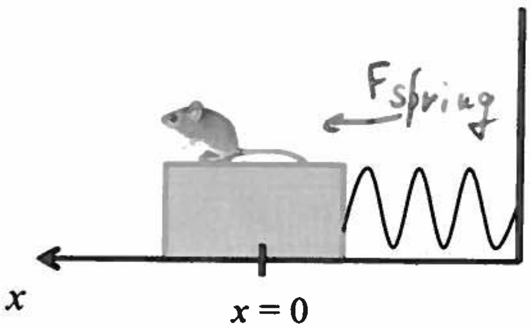
$$m_n v_n = u_2 \left(\frac{M_2 \sin\theta_2 \cos\theta_1}{\sin\theta_1} + M_2 \cos\theta_2 \right)$$

$$\boxed{u_2 = \frac{m_n v_n}{M_2 \left(\frac{\sin\theta_2 \cos\theta_1}{\sin\theta_1} + \cos\theta_2 \right)}}$$

Problem 3 (15 points)

A mouse of mass m_0 is sitting on the block of mass m which is attached to a spring of spring constant k. At time moment $t = 0$ the mouse jumps off the block with the velocity v_0 directed to the left. There is no friction between the block and the table.

a) Find the position of the block as a function of time after the jump.



$$P_x(\text{before}) = P_x(\text{after})$$

$$0 = m_0 v_0 + m u_x$$

$$\boxed{u_x = -\frac{m_0}{m} v_0}$$

$$F_x = m a_x$$

$$-kx = m a_x = m \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x(t=0) = A = 0$$

$$\frac{dx}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\frac{dx}{dt}(t=0) = B\omega = u_x$$

$$\boxed{B = -\frac{m_0}{m\omega} v_0}$$

$$x(t) = B\sin\omega t$$

$$\frac{d^2x}{dt^2} = -B\omega^2\sin\omega t$$

$$-B\omega^2\sin\omega t + \frac{k}{m} B\sin\omega t = 0$$

$$\boxed{\omega^2 = \frac{k}{m}} \quad \boxed{x(t) = -\frac{m}{m_0\omega} v_0 \sin\sqrt{\frac{k}{m}}\,t}$$

b) How long will it take for the block to return to the equilibrium position $x = 0$?

$$x = 0 \quad \sin\sqrt{\frac{k}{m}}\,t = 0; \quad \sqrt{\frac{k}{m}}\,t = \pi; \quad \boxed{t = \pi\sqrt{\frac{m}{k}}} \quad \text{or} \quad \frac{T}{2} = \frac{2\pi}{2\omega}$$

c) Find the total mechanical energy of the block. Show that it does not depend on time.

$$ME = U + KE; \quad U = \frac{kx^2}{2}, \quad KE = \frac{mv^2}{2}$$

$$x(t) = B\sin\omega t, \quad v(t) = B\omega\cos\omega t$$

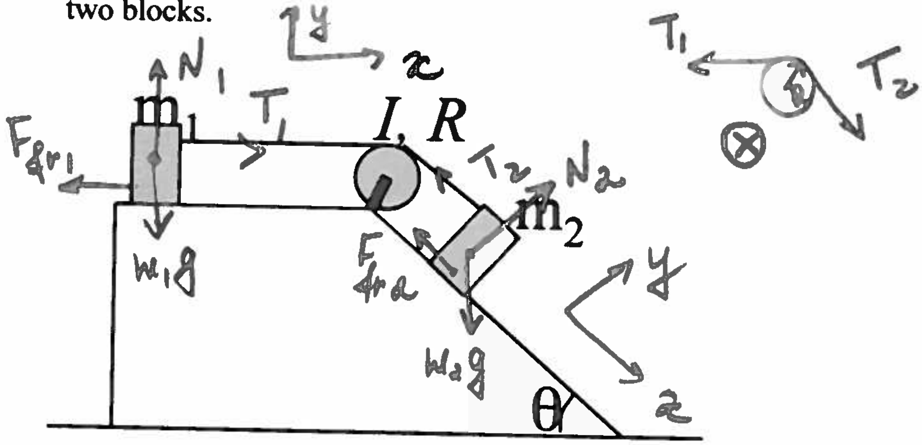
$$ME = \frac{kB^2\sin^2\omega t}{2} + \frac{m}{2} B^2\omega^2\cos^2\omega t; \quad \omega^2 = \frac{k}{m}$$

$$ME = \frac{k}{2} B^2 (\sin^2\omega t + \cos^2\omega t) = \boxed{\frac{k}{2} B^2} = \frac{m_0^2 v_0^2}{2m}$$

Problem 4: (15 points)

Two blocks of masses m_1 and m_2 are connected by a massless unstretchable string over a pulley in the shape of a solid disk of radius R and moment of inertia I around its axis. At $t = 0$ the blocks start moving on a fixed wedge of angle θ, with block m_2 sliding down. Assume that the pulley rotates without friction and there is no slip of the string. The coefficient of friction between the blocks and the surface is μ for both blocks.

a) In the box below write the system of equations that can be solved to find the acceleration of the two blocks.



$$F_x = m a_x$$
$$F_y = m a_y$$
$$\vec{\tau}_{ext} = I\alpha$$
$$a_{cm} = R\alpha$$
$$a_1 = a_2$$

$$\boxed{\begin{aligned} F_{fr_1} &= \mu N_1 \\ F_{fr_2} &= \mu N_2 \end{aligned}}$$

$$\boxed{\begin{aligned} &T_1 - \mu N_1 = m_1 a_x \\ &N_1 - m_1 g = 0 \\ &m_2 g \sin\theta - \mu N_2 - T_2 = m_2 a_x \\ &N_2 - m_2 g \cos\theta = 0 \\ &RT_2 - RT_1 = I\alpha \\ &a_x = R\alpha \end{aligned}}$$

b) Neglect friction. Find the acceleration of the two blocks.

$$\begin{cases} (1) & T_1 = m_1 a_x \\ (2) & m_2 g \sin\theta - T_2 = m_2 a_x \\ (3) & R(T_2 - T_1) = I\alpha \\ (4) & a_x = R\alpha \end{cases}$$

$(1) + (2)$: $T_1 + m_2 g \sin\theta - T_2 = (m_1 + m_2) a_x$

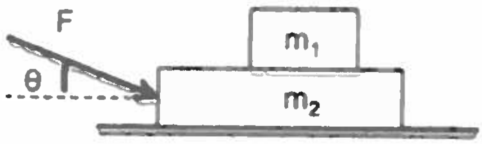
$$m_2 g \sin\theta - \frac{I a_x}{R^2} = (m_1 + m_2) a_x$$

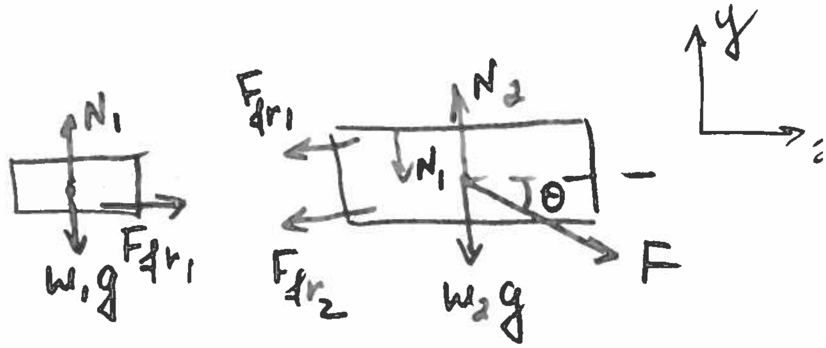
$$\boxed{a_x = \frac{m_2 g \sin\theta}{\frac{I}{R^2} + m_1 + m_2}}$$

Problem 5: (18 points)

Two blocks of masses m_1 and m_2 are placed as shown. There is coefficient of friction μ between all surfaces. A force F is applied to block 2 at an angle θ to the surface as shown.

a) Draw the free-body diagram for both blocks.





b) What is the maximum magnitude F_{max} of the force at which the two blocks still move together with the same acceleration?

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_{fr_1} = \mu N_1 \,; \quad F_{fr_2} = \mu N_2$$
$$\begin{cases} \mu N_1 = m_1 a_x \\ N_1 - m_1 g = 0 \\ F_{max}\cos\theta - \mu N_1 - \mu N_2 = m_2 a_x \\ N_2 - m_2 g - N_1 - F_{max}\sin\theta = 0 \end{cases}$$

$$N_2 = m_2 g + m_1 g + F_{max}\sin\theta$$
$$F_{max}\cos\theta - \mu m_1 g - \mu m_2 g - \mu m_1 g - \mu F_{max}\sin\theta = m_2 a_x = m_2 \mu g$$
$$\mu m_1 g = m_1 a_x \Rightarrow a_x = \mu g$$
$$\boxed{F_{max} = \frac{2\mu(m_2 + m_1)g}{\cos\theta - \mu\sin\theta}}$$

c) Now assume that the force has a known magnitude $F < F_{max}$. In the box below write the system of equations that can be solved to find the acceleration of the blocks and forces of friction between all surfaces.

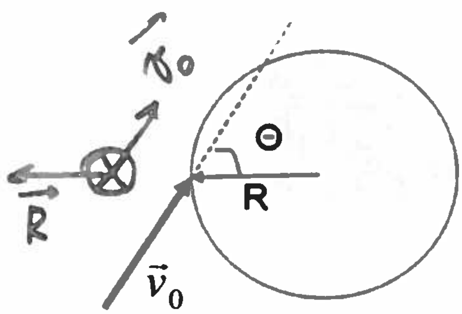
$$F_{fr_1} = m_1 a_x$$
$$N_1 - m_1 g = 0$$
$$F\cos\theta - F_{fr_1} - F_{fr_2} = m_2 a_x$$
$$N_2 - m_2 g - N_1 - F\sin\theta = 0$$
$$F_{fr_2} = \mu N_2$$

Problem 6: (15 points)

A disk-shaped merry-go-round has radius R and moment of inertia I about a vertical axle through its center, and it turns with negligible friction. A child of mass m jumps on the edge of the merry-go-round with horizontal velocity of magnitude v_0, directed as shown. The merry-go-round was initially at rest.

a) Find the magnitude and direction of the angular velocity of the merry-go-round after the child jumps on it.

Top view



$$L(\text{before}) = L(\text{after})$$

$$R m v_0 \sin\theta = m R^2 \omega + I\omega$$

$$\omega = \frac{R m v_0 \sin\theta}{I + mR^2} \quad (\text{clockwise})$$

b) Now assume that as the merry–go-round rotates it experiences a friction force, which creates a torque equal to τ_0 with respect to the vertical axis of rotation. How much time will it take for the merry-go-round to come to a full stop after the jump?

$$\vec{\tau}_{ext} = I^{tot}\alpha \ (rhr) \quad \text{or} \quad \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

$$-\tau_0 = (I + mR^2)\,\alpha$$

$$\alpha = -\frac{\tau_0}{I + mR^2}$$

$$\omega(t) = \int \alpha(t)\,dt = -\frac{\tau_0}{I + mR^2}t + \omega_0$$

$$-\frac{\tau_0}{I + mR^2}t^* + \omega_0 = 0 \ ; \quad \boxed{t^* = \frac{\omega_0(I + mR^2)}{\tau_0}}$$

$$\boxed{\omega_0 = \frac{R m v_0 \sin\theta}{I + mR^2}}$$

$$\boxed{t^* = \frac{R m v_0 \sin\theta}{\tau_0}}$$

Problem 7: (7 points)

The motion of a small object of mass m is observed in the region along the x axis between $x = 0$ and $x = L$. The kinetic energy is measured and found to vary with its position according to

$$KE(x) = K_0(1 - \frac{x^2}{L^2})$$

where K_0 is a known constant. Assuming that only a conservative force acts on the object, what is the force?

$$KE + U = \text{Const}$$

$$U = \text{Const} - KE$$

$$F = -\frac{dU}{dx} = \frac{dKE}{dx} = \boxed{-K_0 \frac{2x}{L^2}}$$