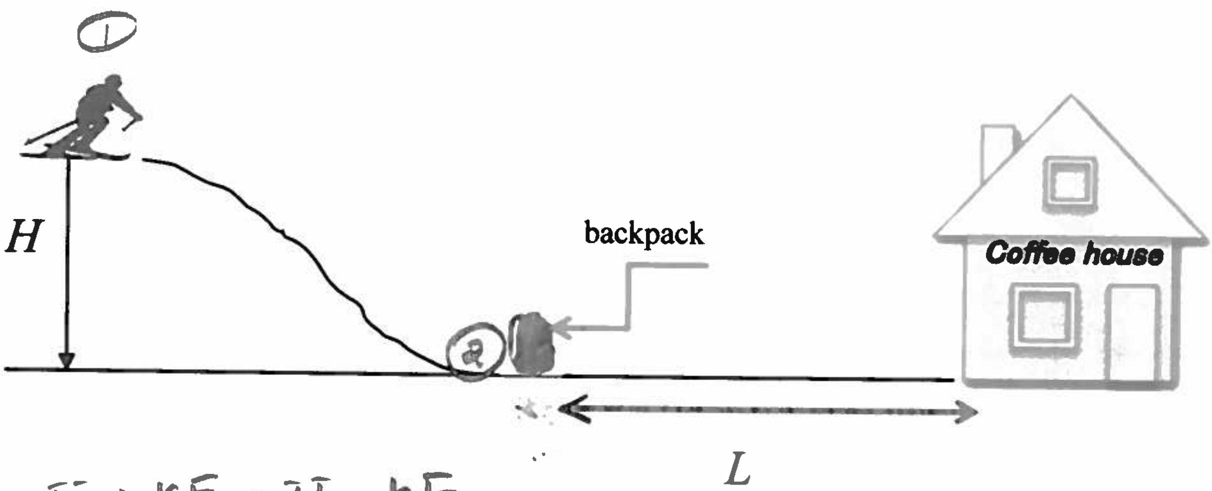
Problem 1: (16 points)

A skier with mass m_1 starts from rest at the top of the frictionless ski slope H meters high. As the skier reaches the horizontal path, he grabs his backpack of mass m_2. After that he skies along a horizontal surface where the coefficient of friction is μ. He stops right in front of the coffee house. Find distance L.



$$U_1 + KE_1 = U_2 + KE_2$$

$$m_1 g H = \frac{m_1 v^2}{2} \; ; \; v = \sqrt{2gH}$$

$$p_x(\text{before}) = p_x(\text{after})$$

$$m_1 v_1 = (m_1 + m_2) u \; ; \; u = \frac{m_1}{m_1 + m_2} \sqrt{2gH}$$

$$W^{net} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \frac{m v_f^2}{2} - \frac{m v_i^2}{2}$$

$$W^{friction} = \int_0^L -\mu N \, dx = -\mu (m_1 + m_2) g L = -\frac{(m_1 + m_2)}{2} \frac{m_1^2 \, 2gH}{(m_1 + m_2)^2}$$

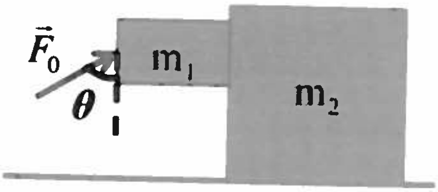
$$L = \frac{m_1^2}{(m_1 + m_2)^2} \frac{H}{\mu}$$

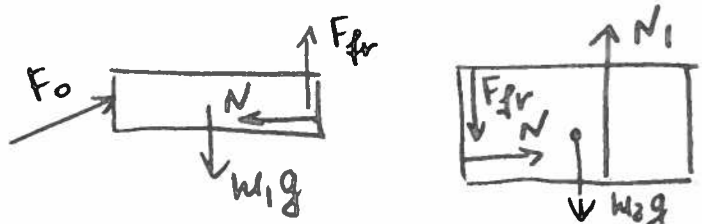
Answer: $$\boxed{L = \left(\frac{m_1}{m_1 + m_2}\right)^2 \frac{H}{\mu}}$$

Problem 2: (16 points)

The two blocks, masses m_1 and m_2 are not attached to each other. The blocks are on frictionless table but the coefficient of friction between the blocks is μ. A force F_0 is applied to the smaller block at angle θ and the blocks are moving together.

a) Draw the free body diagram for both blocks. Assume that force F_0 is small enough so that the force of friction acting on the block m_1 is up, but F_0 is large enough so that the small block does not slip down.





b) In the box below write the system of equations that can be solved to find the acceleration of the blocks and force of friction between the blocks. Don't forget to indicate the coordinate system. Find the acceleration of the blocks and force of friction between them.

$$\Sigma F_x = m a_x$$

$$\Sigma F_y = m a_y$$

$$F_0 \sin\theta - N = m_1 a_x$$

$$F_{fr} - m_1 g + F_0 \cos\theta = 0$$

$$N = m_2 a_x$$

$$F_0 \sin\theta = (m_1 + m_2) a_x$$

$$a_x = \frac{F_0 \sin\theta}{m_1 + m_2}$$

$$F_{fr} = m_1 g - F_0 \cos\theta$$

Answer: $a_x = \dfrac{F_0 \sin\theta}{m_1 + m_2}$

$F_{friction} = m_1 g - F_0 \cos\theta$

c) What is the minimum magnitude of force $\vec{F}_0$ that must be applied to keep the small block from slipping down the larger block?

$$\begin{cases} F_0 \sin\theta - N = m_1 a_x \\ \mu N - m_1 g + F_0 \cos\theta = 0 \\ N = m_2 a_x \end{cases}$$

Or $F_{fr} = m_1 g - F_0 \cos\theta = \mu N$

$$N = m_2 \frac{F_0 \sin\theta}{m_1 + m_2}$$

$$m_1 g - F_0 \cos\theta = \mu \frac{m_2}{m_1 + m_2} F_0 \sin\theta$$

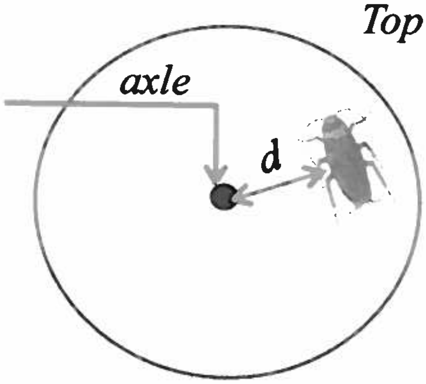
Answer: $F_0 = \dfrac{m_1 g}{\mu \frac{m_2}{m_1 + m_2} \sin\theta + \cos\theta}$

Problem 3: (15 points)

A lazy Susan (a circular disk mounted on a vertical axle) is at rest at $t = 0$. It has frictionless bearings. A Texas sized cockroach of mass m starts running counterclockwise around a circle distance d from the center with velocity of v_0 relative to the ground. As a result, the lazy Susan starts spinning with angular velocity ω_0.

a) Find the direction of spinning and moment of inertia of the lazy Susan about the central axis (note that mass of lazy Susan and its radius are not given).

Top view



$$\vec{L}_{initial} = \vec{L}_{final}$$

$$0 = m v_0 d - I\omega_0$$

$$\boxed{I = \frac{m v_0 d}{\omega_0}}$$

clockwise spinning

b) The cockroach finds a bread crumb and stops. What is the angular velocity of the lazy Susan after the cockroach stops?

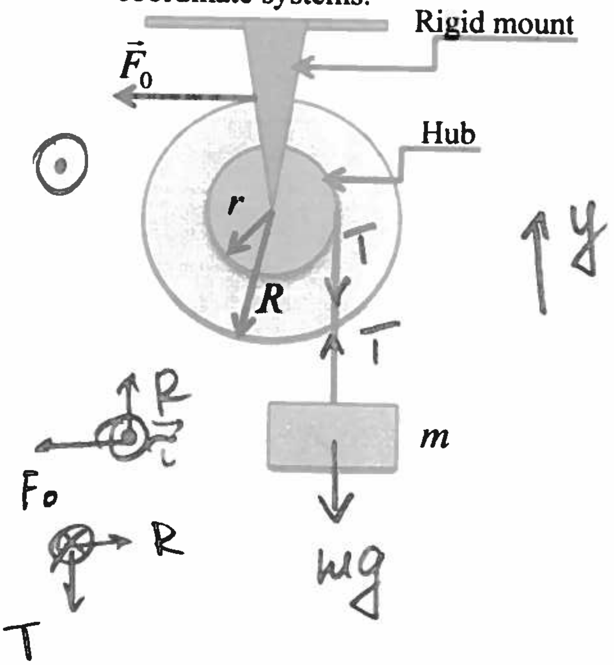
$$\vec{L}_{initial} = \vec{L}_{final}$$

$$\underbrace{m v_0 d - I\omega_0}_{=0} = (-I - md^2)\,\omega$$

$$\boxed{\omega = 0}$$

Problem 4: (15 points)

A yo-yo-shaped device mounted on a horizontal frictionless axis is used to lift a box of mass m. The outer radius of the device is R, and the radius of the hub is r. When a constant horizontal force, $\vec{F}_0$ is applied to a rope wrapped around the outside of the device, the box, which is suspended from a rope wrapped around the hub, is moving up. The moment of inertia of the device about its axis of rotation is I. The ropes do not slip. a) In the box below write the system of equations that can be solved to find the acceleration of the block. The problem will not be graded without free body diagrams and coordinate systems.



$$\vec{\tau}_{ext} = I\alpha \ (rhr)$$

$$F_y = m a_y$$

$$\boxed{\begin{aligned} T - mg &= m a_y \\ F_0 R - T r &= I\alpha \\ a_y &= r\alpha \end{aligned}}$$

b) Find the acceleration of the block.

$$F_0 R - Tr = I \frac{a_y}{r}$$

$$T = \frac{F_0 R - I\frac{a_y}{r}}{r} = F_0 \frac{R}{r} - \frac{I a_y}{r^2}$$

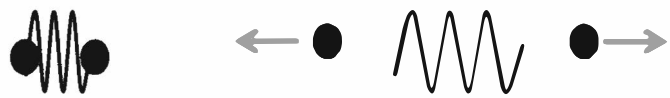
$$F_0 \frac{R}{r} - \frac{I a_y}{r^2} - mg = m a_y$$

$$a_y \left(\frac{I}{r^2} + m\right) = F_0 \frac{R}{r} - mg$$

$$a_y = \frac{F_0 \frac{R}{r} - mg}{m + \frac{I}{r^2}}$$

Problem 5: (14 points)

Particle A and particle B are held together with a compressed spring between them. When they are released, the spring pushes them apart, and they then fly off in opposite directions, free of the spring. The mass A is 2 times the mass of B, and the energy originally stored in the spring is E_0. Assume that the spring has negligible mass and that all its stored energy is transferred to the particles. Once transfer is complete, what are the kinetic energies of both particles? Clearly indicate the laws that you use. Hint: first find the ratio of velocities and the ratio of kinetic energies of the particles.



Before After

$$U_1 + KE_1 = U_2 + KE_2$$

$$p_x(\text{before}) - p_x(\text{after})$$

$$\begin{cases} E_0 = KE_A + KE_B \\ m_A = 2m_B \\ 0 = -m_A v_A + m_B v_B \end{cases} \Rightarrow v_A = \frac{m_B}{m_A} v_B = \frac{v_B}{2}$$

$$\frac{KE_A}{KE_B} = \frac{m_A v_A^2 2}{2 m_B v_B^2} = \frac{1}{2}$$

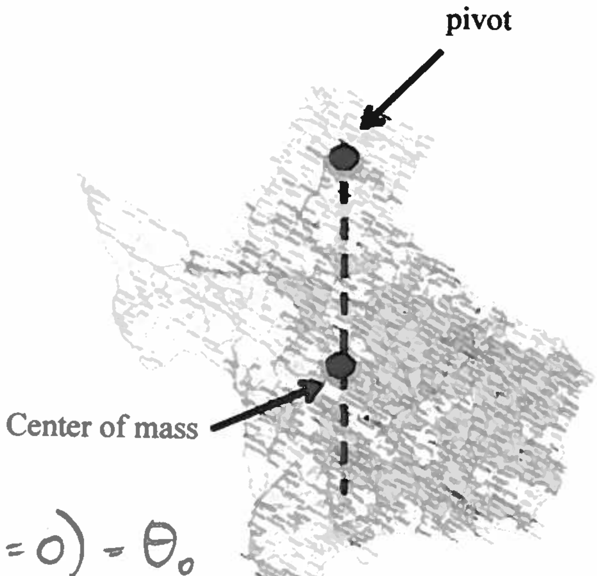
$$E_0 = \frac{KE_B}{2} + KE_B = \frac{3}{2} KE_B$$

$$KE_B = \frac{2}{3} E_0 ; \quad KE_A = \frac{E_0}{3}$$

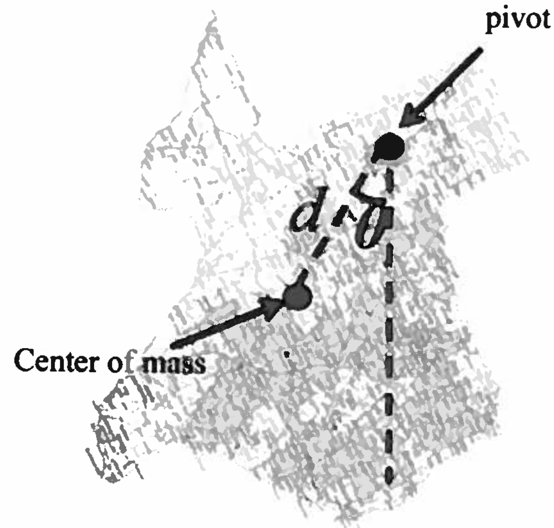
Answer: $KE_A = \frac{2E_0}{3}$; $KE_B = \frac{1}{3} E_0$

Problem 6: (15 points)

An object of mass M shaped as a map of Texas is pivoted at a point that is a distance d from the center of mass. It has moment of inertia I with respect to the horizontal axis going through the pivot point perpendicular to the plane of the map. At $t = 0$ it is displaced from equilibrium by an angle θ_0 and released from rest. (a) Derive the equation of motion for an angle θ and show that for small angles ($\sin\theta \approx \theta$) the object undergoes simple harmonic motion. (b) Solve it for angle θ as a function of time. (c) Find when the object returns to an initial position for the first time.



equilibrium



out of equilibrium

$$\theta(t=0) = \theta_0$$

$$\frac{d\theta}{dt}(t=0) = 0$$

$$\vec{\tau}_{ext} = I\alpha \ (rhr)$$

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0$$

$$\theta(t) = A\cos\omega t + B\sin\omega t$$

$$\frac{d\theta}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\theta(t=0) = A = \theta_0$$

$$\frac{d\theta}{dt}(t=0) = B\omega = 0 \rightarrow B = 0$$

$$\boxed{\theta(t) = \theta_0\cos\omega t}$$

$$\frac{d^2\theta}{dt^2} = -A\omega^2\cos\omega t$$

$$-A\omega^2\cos\omega t + \frac{mgd}{I}A\cos\omega t = 0$$

$$\boxed{\omega^2 = \frac{mgd}{I}}$$

$$T = \frac{2\pi}{\omega}$$

$$\boxed{T = \frac{2\pi\sqrt{I}}{\sqrt{mgd}}}$$

Problem 7: (14 points)

A single conservative force $F(x)$ acts on a particle of mass m that moves along an x axis. The potential energy $U(x)$ associated with $F(x)$ is given by

$$U(x) = -U_0 \frac{x}{x_0} e^{-x/x_0}$$

Where x_0 and U_0 are known positive constants. At $x = x_0/2$ the particle has a kinetic energy of KE_0 and moves in a positive x direction.

a) What is the mechanical energy of the system?

$$ME = -U_0 \frac{x_0}{2x_0} e^{-\frac{x_0}{2x_0}} + KE_0 = -\frac{U_0}{2} e^{-\frac{1}{2}} + KE_0$$

b) Determine value of x at which the maximum kinetic energy occurs.

KE is max when U is min

$$\frac{dU}{dx} = -\frac{U_0}{x_0} e^{-\frac{x}{x_0}} - \frac{U_0}{x_0} x e^{-\frac{x}{x_0}} \left(-\frac{1}{x_0}\right) = 0$$

$$1 - \frac{x}{x_0} = 0$$

$$\boxed{x = x_0}$$

c) Suppose instead the potential energy is given by $U(x) = -U_0 \frac{x}{x_0}$. At $t = 0$ the particle was at $x = x_0/2$ and had a velocity v_0.

Find the velocity and position of the particle as a function of time.

$$F_x = -\frac{\partial U}{\partial x} = \frac{U_0}{x_0}$$

$$F_x = ma_x ; \quad a_x = \frac{U_0}{m x_0}$$

$$v(t) = \frac{U_0}{m x_0} t + v_0$$

$$x(t) = \frac{1}{2} \frac{U}{m x_0} t^2 + v_0 t + \frac{x_0}{2}$$