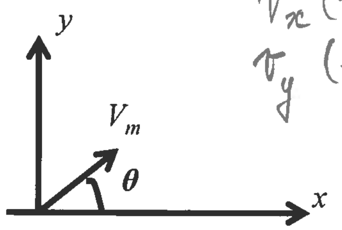
Problem 1: (15 points)

a) A cannon ball of mass m is fired with initial velocity V_m at an angle θ with the horizontal. What does V_m have to be so that the cannon ball will reach a maximum of H if the angle θ is known?



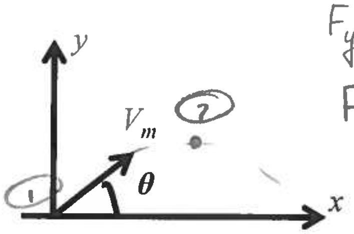
$$v_x(t) = v_m \cos\theta$$
$$v_y(t) = -gt + v_m \sin\theta$$
$$v_y(t^*) = -gt^* + v_m \sin\theta = 0$$
$$t^* = \frac{v_m \sin\theta}{g};$$

$$y(t) = -\frac{1}{2} g t^2 + v_m \sin\theta \, t$$
$$y(t^*) = -\frac{1}{2} g \frac{v_m^2 \sin^2\theta}{g^2} + \frac{v_m^2 \sin^2\theta}{g} =$$
$$= \frac{1}{2} \frac{v_m^2 \sin^2\theta}{g} = H$$
$$v_m = \frac{1}{\sin\theta}\sqrt{2gH}$$

Answer:
$$v_m = \frac{\sqrt{2gH}}{\sin\theta}$$

Or
$$\frac{m v_m^2}{2} = mgH + \frac{m}{2} v_m^2 \cos^2\theta$$
$$\frac{m}{2} v_m^2 (1 - \cos^2\theta) = mgH$$
$$v_m = \frac{\sqrt{2gH}}{\sin\theta}$$

b) A cannon ball of mass m is fired with initial velocity V_m at an angle θ with the horizontal. In addition to gravity there is a strange force on the cannon ball that is in the same direction as gravity but has a magnitude that increases with height according to βy^2. What does V_m have to be so that the cannon ball will reach a maximum of H if the angle θ is known?

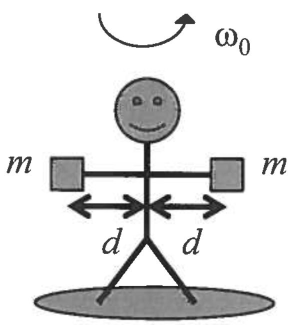


$$F_y = -mg - \beta y^2$$
$$F_y = -\frac{\partial U}{\partial y}; \quad U = -\int F_y \, dy = -\int(-mg - \beta y^2)\, dy =$$
$$= mgy + \frac{\beta y^3}{3} + Const$$
$$U_1 + KE_1 = U_2 + KE_2$$
$$\frac{m v_m^2}{2} = mgH + \frac{\beta H^3}{3} + \frac{m v_m^2 \cos^2\theta}{2}$$
$$\frac{m v_m^2}{2} \underbrace{(1 - \cos^2\theta)}_{= \sin^2\theta} = mgH + \frac{\beta H^3}{3}; \quad v_m^2 = \frac{2\left(mgH + \frac{\beta H^3}{3}\right)}{m \sin^2\theta}$$

Answer:
$$v_m = \sqrt{\frac{2\left(mgH + \frac{\beta H^3}{3}\right)}{m \sin^2\theta}}$$

Problem 2: (15 points)

In a famous Physics 218 experiment a student stands on a platform which is free to rotate on frictionless bearings. The moment of inertia of the platform about the vertical axle through its center is a known constant I. He has his arms extended with a huge mass m in each hand. If he is set into rotation with angular velocity ω_0 and then drops his hands to his sides, what happens to his angular velocity? Assume that the man's mass is negligible and that his arms have length d when extended and are $d/4$ from the center of his body when at his sides.

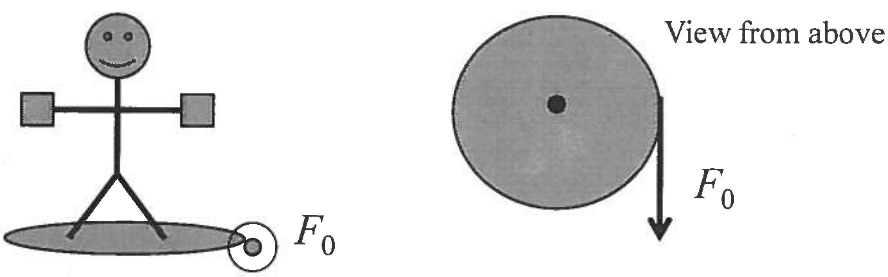


$$\vec{L}_{initial} = \vec{L}_{final}$$

$$2md^2\omega_0 + I\omega_0 = 2m\left(\frac{d}{4}\right)^2\omega_f + I\omega_f$$

$$\boxed{\omega_f = \frac{(2md^2 + I)\,\omega_0}{2m\left(\frac{d}{4}\right)^2 + I}}$$

Call the angular velocity that you found ω_f. Now the instructor applies a constant decelerating force F_0 to the edge of the platform that is directed as shown. How long will it take for the platform to come to stop after the force is applied? The radius of the platform is R.



$$\vec{\tau}_{ext} = I_{tot}\,\alpha \quad (rhr)$$

$$F_0 R = I_{tot}\,\alpha \;, \quad I_{tot} = I + 2\left(\frac{d}{4}\right)^2 m$$

$$\alpha = \frac{F_0 R}{I_{tot}}$$

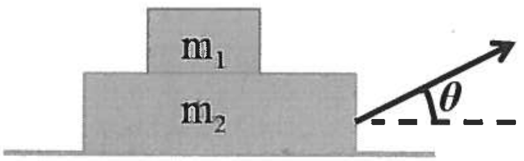
$$\omega = \int \alpha\, dt = \int \frac{F_0 R}{I_{tot}}\, dt = \frac{F_0 R}{I_{tot}}\, t - \omega_f$$

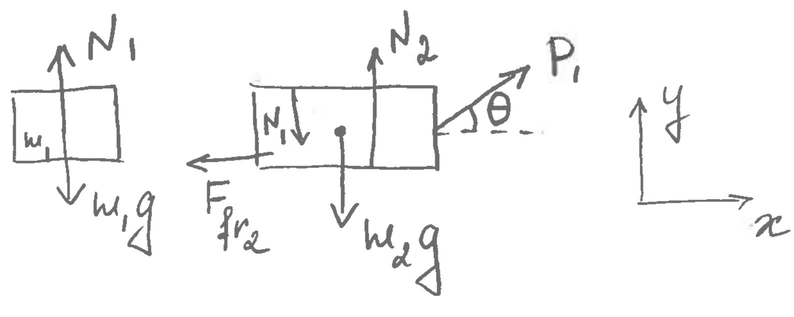
$$\frac{F_0 R}{I_{tot}}\, t^* - \omega_f = 0$$

$$\boxed{t^* = \frac{I_{tot}\,\omega_f}{F_0 R}}$$

Problem 3: (15 points)

a) Two blocks, masses m_1 and m_2 are pulled as shown. The coefficient of friction between the blocks is μ_1, and the coefficient of friction between the lower block and the floor is μ_2. The pulling force of magnitude P_1 is such that they are moving with constant velocity. The angle θ is known and is the same in all parts of this problem. Draw the free body diagram for both blocks. Find the magnitude of the pulling force P_1.





$F_x = m a_x \qquad F_y = m a_y$

$$\begin{cases} P_1 \cos\theta - \mu_2 N_2 = 0 \\ N_2 - m_2 g - N_1 + P_1 \sin\theta = 0 \\ N_1 - m_1 g = 0 \end{cases}$$

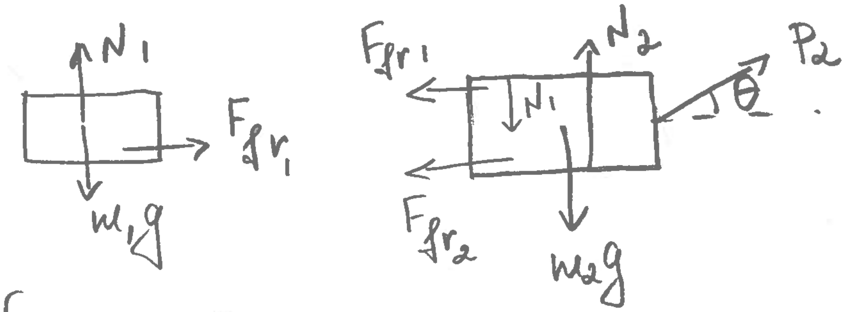
$N_2 = m_2 g + m_1 g - P_1 \sin\theta$

$P_1 \cos\theta - \mu_2\left((m_1+m_2)g - P_1 \sin\theta\right) = 0$

$P_1(\cos\theta + \mu_2 \sin\theta) = \mu_2 (m_1 + m_2) g$

Answer: $P_1 = \dfrac{\mu_2 (m_1+m_2) g}{\cos\theta + \mu_2 \sin\theta}$

b) The magnitude of the pulling force is increased so that the blocks accelerate. Draw the free body diagram for both blocks. Find the maximum acceleration before the top block starts slipping.



$$\begin{cases} \mu_1 N_1 = m_1 a_x \\ N_1 - m_1 g = 0 \end{cases}$$

$\mu_1 m_1 g = m_1 a_x$

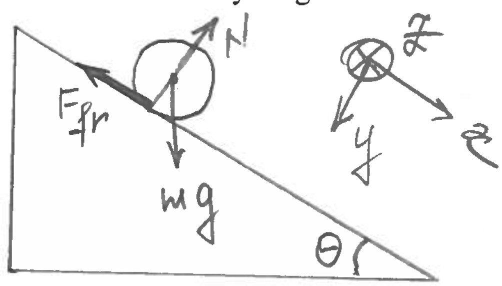
$a_x = \mu_1 g$

Answer: $a_x = \mu_1 g$

Problem 4: (15 points)

A solid cylinder of mass M is placed on the top of an inclined plane of height H and inclination angle θ. It starts rolling down without slipping from rest. The coefficient of friction between the cylinder and the incline is μ. (Reminder: a cylinder of mass M and radius R has a moment of inertia $I = (1/2)\,MR^2$ about the central axis.)

a) Draw the free body diagram.



b) Find the acceleration of the cylinder and the force of friction between the cylinder and the incline.

$$\begin{cases} F_x = ma_x \\ F_y = ma_y \\ \vec{\tau}_{ext} = I(\alpha)(rhr) \end{cases}$$

$$\begin{cases} Mg\sin\theta - F_{fr} = Ma_{cm_x} \\ F_{fr}R = I\alpha \\ a_{cm_x} = R\alpha \end{cases}$$

$$\begin{cases} F_{fr}R = \dfrac{MR^2}{2}\dfrac{a_x}{R} \\ F_{fr} = Mg\sin\theta - Ma_x \end{cases}$$

$$Mg\sin\theta - Ma = \frac{Ma_x}{2}$$

$$g\sin\theta = a_x + \frac{a_x}{2} = \frac{3}{2}a_x$$

$$a_{cm_x} = \frac{2}{3}g\sin\theta$$

$$F_{fr} = \frac{M}{2}a_x = \frac{Mg\sin\theta}{3}$$

Answer:

$a_x = \frac{2}{3}g\sin\theta$

$F_{fr} = \dfrac{Mg\sin\theta}{3}$

c) Find the maximum angle of the incline right before the cylinder starts slipping.

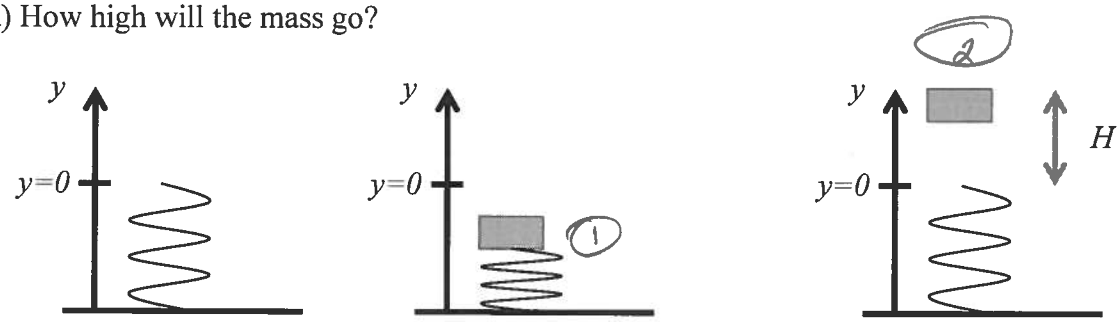
$$\begin{cases} F_{fr} = \mu N = \mu Mg\cos\theta = \frac{1}{3}Mg\sin\theta \\ -N + Mg\cos\theta = 0 \end{cases}$$

$$\boxed{\tan\theta = 3\mu}$$

Problem 5: (15 points)

A vertical ideal spring, spring constant k, is compressed a distance A. A mass m is placed on top of the spring and then released.

a) How high will the mass go?



$$U_1 + KE_1 = U_2 + KE_2$$

$$\frac{kA^2}{2} - mgA = mgH$$

$$\boxed{H = \left(\frac{kA^2}{2} - mgA\right)\frac{1}{mg}}$$

Answer:

b) If instead the force exerted by the spring is given by $F_y = -(ky+b)$, how high will the mass go?

$$W_{gravity} = -mg(H+A) \qquad W_{net} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} \; 0$$

$$W_{spring} = \int_{-A}^{0}(-ky-b)\,dy = -\frac{ky^2}{2} - by\Big|_{-A}$$

$$= \frac{kA^2}{2} - bA$$

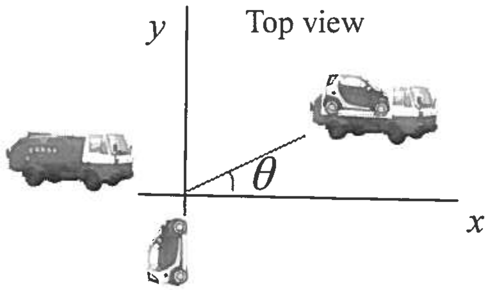
$$-mg(H+A) + \frac{kA^2}{2} - bA = 0$$

$$\boxed{H = \frac{1}{mg}\left(\frac{kA^2}{2} - bA\right) - A}$$

Answer:

Problem 6: (15 points)

A small car of mass m_1 is traveling due north when it collides with a pick-up truck of mass m_2 which was traveling due east. After the collision the two vehicles move off together at an angle θ north of east (in the horizontal x-y plane). The driver of the car claimed that the truck driver was at fault because he was exceeding the speed limit, going with a velocity v_2. If this were true, what was the car's initial velocity?



p_x (before) = p_x (after)

p_y (before) = p_y (after)

$m_2 v_2 = (m_1 + m_2) u \cos\theta \quad (1)$

$m_1 v_1 = (m_1 + m_2) u \sin\theta \quad (2)$

$(2):(1) \quad \frac{m_1 v_1}{m_2 v_2} = \tan\theta$

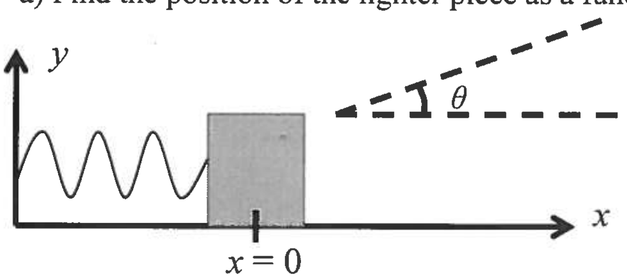
$$\boxed{v_1 = \frac{m_2}{m_1} v_2 \tan\theta}$$

Answer:

Problem 7: (15 points)

A small block of mass m is at rest on a frictionless surface. It is attached to a spring with spring constant k. In an instantaneous explosion it explodes into two pieces. The heavier piece, mass $\frac{2}{3}m$, goes off at a known angle θ with a velocity of magnitude v_2 as shown. The lighter piece goes off to the left.

a) Find the position of the lighter piece as a function of time.



$$p_x(\text{before}) = p_x(\text{after})$$

$$0 = \frac{2}{3} m v_2 \cos\theta + \frac{1}{3} m v_{1x}$$

$$\boxed{v_{1x} = -2 v_2 \cos\theta}$$

$$x(t=0) = 0$$
$$v(t=0) = v_{1x}$$

$$-kx = \frac{m_1}{3} \frac{d^2 x}{dt^2}$$

$$\frac{d^2 x}{dt^2} + \frac{3k}{m_1} x = 0$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x(t=0) = \boxed{A = 0}$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$v(t=0) = B\omega = -2 v_2 \cos\theta$$

$$\boxed{B = -\frac{2 v_2 \cos\theta}{\omega}}$$

$$x(t) = B\sin\omega t$$

$$\frac{dx}{dt} = B\omega\cos\omega t$$

$$\frac{d^2x}{dt^2} = -B\omega^2 \sin\omega t$$

$$-B\omega^2 \sin\omega t + \frac{3k}{m_1} B\omega\sin\omega t = 0$$

$$\omega^2 = \frac{3k}{m_1}$$

$$x(t) = -\frac{2 v_2 \cos\theta}{\omega} \sin\omega t$$

$$\omega = \sqrt{\frac{3k}{m_1}}$$

b) How long will it take for the lighter block to come to the point $x = 0$?

$$x = 0 \Rightarrow \sin\omega t^* = 0 \Rightarrow \omega t^* = \pi$$

$$\boxed{t^* = \frac{\pi}{\omega}}$$