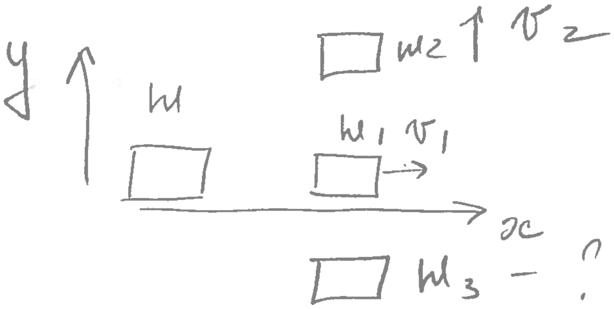
Problem 1: (15 points)

A block of mass m is sliding on a frictionless table with a velocity of magnitude v_0. (Call this $+x$ direction). It splits into three pieces after an internal explosion: one with mass m_1 goes off in $+x$ direction with velocity of magnitude v_1. A second piece, mass m_2, goes off perpendicular to the original direction, but still in the plane of the table, with velocity of magnitude v_2. (Call this the $+y$ direction). Find the velocity of the third piece.



p_x (before) = p_x (after)

p_y (before) = p_y (after)

$$\begin{cases} m v_0 = m_1 v_1 + m_3 v_{3x} \\ 0 = m_2 v_2 + m_3 v_{3y} \end{cases}$$

$$m_3 = m - (m_1 + m_2)$$

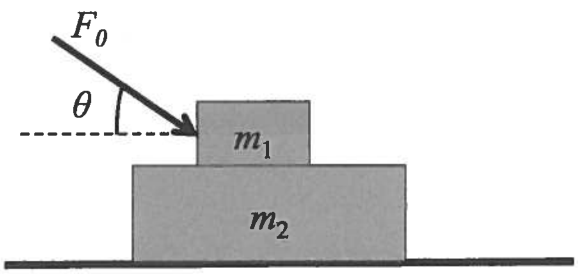
$$v_{3x} = \frac{m v_0 - m_1 v_1}{m - (m_1 + m_2)}$$

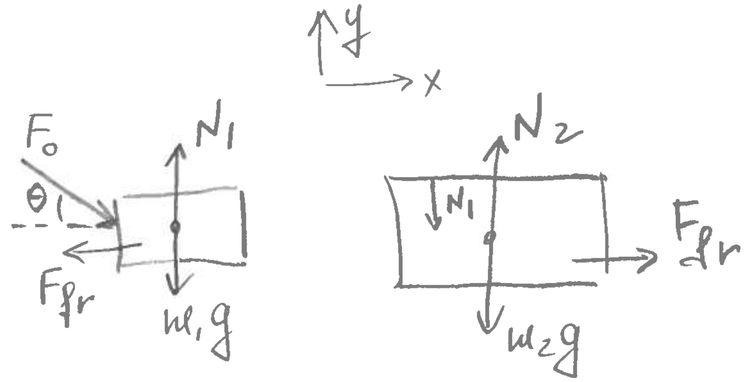
$$v_{3y} = - \frac{m_2 v_2}{m - (m_1 + m_2)}$$

Problem 2: (20 points)

Two blocks of masses m_1 and m_2 are placed as shown. There is a force of friction between the blocks. The block of mass m_2 is on a frictionless surface. A force F_0 is applied to the block of mass m_1 at an angle θ to the surface as shown. The blocks are moving together without slipping.

a) Draw the free-body diagram for both blocks.





b) In the box below, write the system of equations that could be solved to find the acceleration of the blocks.

$$F_x = m a_x$$
$$F_y = m a_y$$

$$F_0 \cos\theta - F_{fr} = m_1 a_x$$
$$-F_0 \sin\theta + N_1 - m_1 g = 0$$
$$F_{fr} = m_2 a_x$$
$$[N_2 - m_2 g - N_1 = 0] \text{ not needed}$$

c) Find the acceleration of the blocks and the forces that the block m_2 exerts on block m_1.

$$F_0 \cos\theta - m_2 a_x = m_1 a_x$$
$$a_x = \frac{F_0 \cos\theta}{m_1 + m_2}$$
$$F_{fr} = \frac{m_2}{m_1 + m_2} F_0 \cos\theta$$

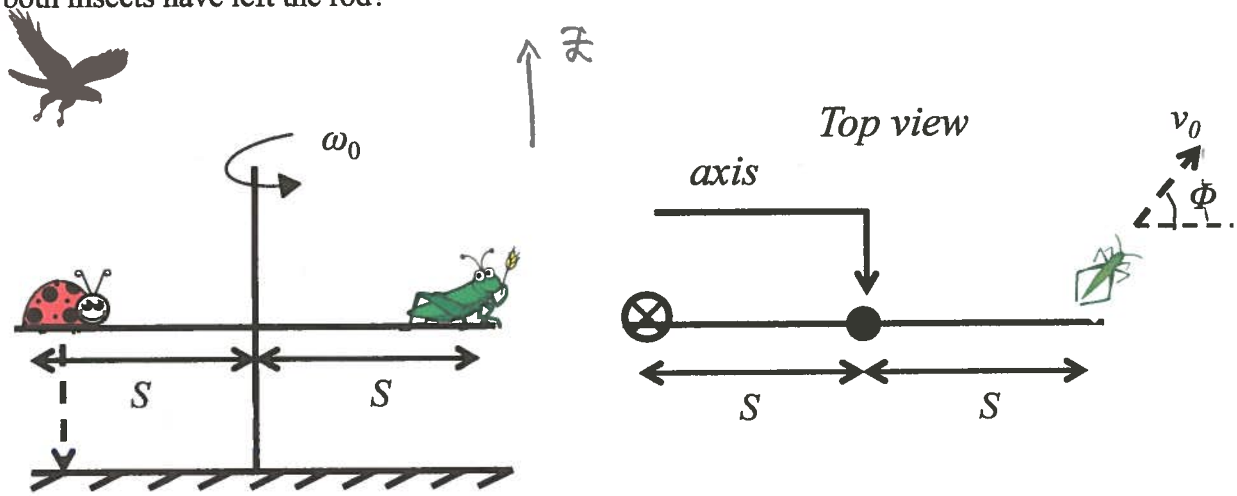
$$N_1 = m_1 g + F_0 \sin\theta$$

Answer:

$$a_x = \frac{F_0 \cos\theta}{m_1 + m_2}; \quad F_{fr} = \frac{m_2}{m_1 + m_2} F_0 \cos\theta; \quad N_1 = m_1 g + F_0 \sin\theta$$

Problem 3 : (15 points)

A bug and a grasshopper enjoy a ride on a spinning rod that rotates freely about a vertical axle with angular velocity ω_0. The bug has mass m_1, the grasshopper has mass m_2, and the rod has moment of inertia I_R about the axis of rotation. Each of the insects is at a distance S from the axle. Suddenly the bug and the grasshopper see an approaching bird. In desperation, the grasshopper jumps off horizontally in the direction shown (see the top view figure) with velocity of magnitude v_0, while the bug falls off the rod vertically down. What is the angular velocity (magnitude and direction) of the rod after both insects have left the rod?



$$L_z(\text{initial}) = L_z(\text{final})$$

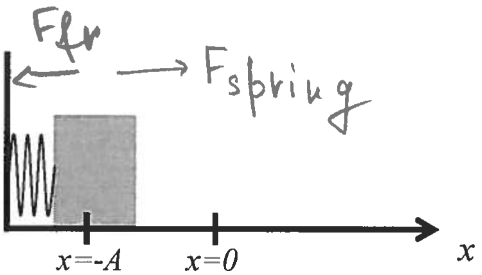
$$m_1 S^2 \omega_0 + m_2 S^2 \omega_0 + I_R \omega_0 = m_2 v_0 \sin\varphi\, S + I_R\, \omega$$

$$\omega = \frac{(m_1 S^2 + m_2 S^2 + I_R)\,\omega_0 - m_2 v_0 \sin\varphi\, S}{I_R} \quad \text{ccw}$$

Problem 4: (20 points)

A block of mass m is attached to a spring, spring constant k. The spring is compressed by amount of A from the point at which the spring is unstretched. The force of the spring is $F_x = -(kx - bx^3)$. Here b is a known constant. The coefficient of friction between the block and the surface is μ.

a) Prove that the force of the spring is a conservative force.



$$F_x = -(kx - bx^3)$$

$$F_x = -\frac{dU}{dx}; \quad U = -\int F_x\,dx =$$

$$= -\int -(kx - bx^3)\,dx = \frac{kx^2}{2} - \frac{bx^4}{4} + \text{const}$$

b) Find the equation that could be solved to find x_s, the point at which the block will stop if released from rest. Do not solve the equation.

$$W_{net} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{-A}^{x_s} \left(-(kx - bx^3) - \mu N\right) dx =$$

$$= -\frac{kx^2}{2} + \frac{bx^4}{4} - \mu mg x \Big|_{-A}^{x_s} =$$

$$F_y = ma_y$$

$$N - mg = 0$$

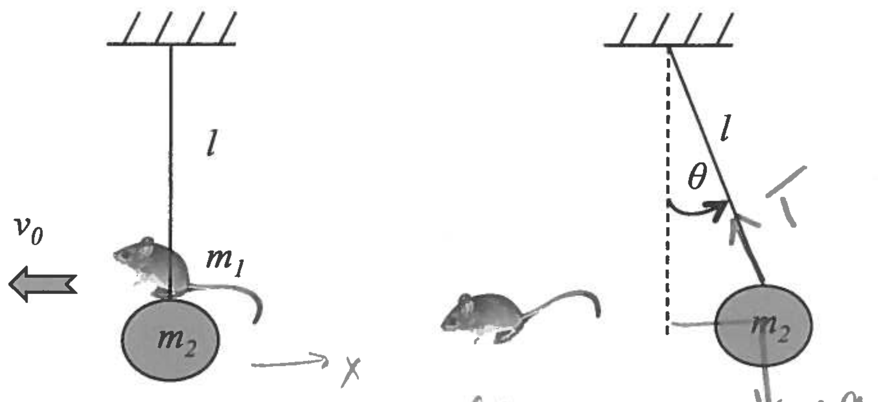
$$-\frac{kx_s^2}{2} + \frac{bx_s^4}{4} - \mu mg x_s + \frac{kA^2}{2} - \frac{bA^4}{4} - \mu mg A =$$

$$= \underbrace{\frac{mv_f^2}{2}}_{=0} - \underbrace{\frac{mv_i^2}{2}}_{=0} = 0$$

Problem 5: (15 points)

A mouse, mass m_1, sits on the bob of the pendulum, mass m_2. The length of the pendulum string is l. The mouse jumps horizontally with a velocity of magnitude v_0. Assume that the mouse jumps so quickly that the string remains vertical during the impact.

a) Find the velocity of the bob right after the mouse's jump.



$$\theta(t=0) = 0$$

$$\frac{d\theta}{dt}(t=0) = \frac{m_1 v_0}{m_2 l}$$

$$p_x(\text{before}) = p_x(\text{after})$$

$$0 = -m_1 v_0 + m_2 v_x; \quad v_x = \frac{m_1}{m_2} v_0$$

b) Derive the equation that could be solved to find the dependence $\theta(t)$, where θ is an angle the string makes with vertical, if the displacement is small ($\sin\theta \approx \theta$).

$$\vec{\tau}_{ext} = I\alpha \ (\text{rhr})$$

$$-\cancel{m_2} g\, l \sin\theta = \cancel{m_2} l^{\cancel{2}} \frac{d^2\theta}{dt^2}$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0}$$

c) Find θ(t). How long will it take for the pendulum to return to the equilibrium?

$$\theta(t) = A\cos\omega t + B\sin\omega t$$

$$\theta(t=0) = \boxed{A = 0}$$

$$\frac{d\theta}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\frac{d\theta}{dt}(t=0) = B\omega = \frac{m_1}{m_2}\frac{v_0}{l}; \quad \boxed{B = \frac{m_1 v_0}{m_2 l \omega}}$$

$$\frac{d^2\theta}{dt^2} = -B\omega^2 \sin\omega t; \quad -\cancel{B}\omega^2 \cancel{\sin\omega t} + \frac{g}{l}\cancel{B}\cancel{\sin\omega t} = 0$$

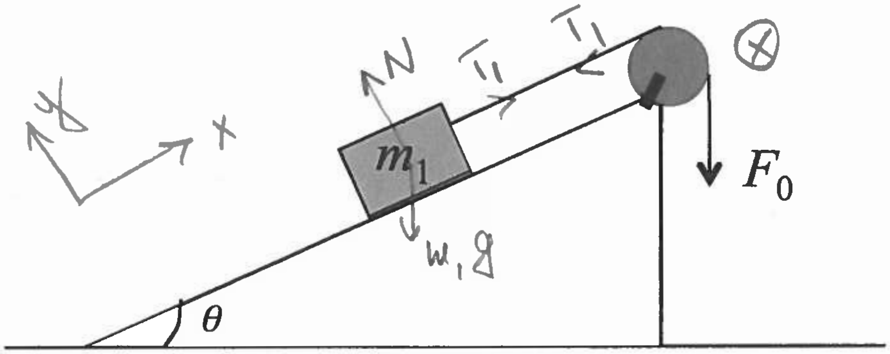
$$\boxed{\omega^2 = \frac{g}{l}} \qquad \boxed{\theta(t) = B\sin\omega t}$$

$$T = \frac{2\pi}{\omega}$$

$$\boxed{t^* = \frac{T}{2} = \frac{\pi}{\omega}}$$

Problem 6: (20 points)

Block of mass m_1 is placed on a frictionless inclined plane of angle θ. A force of magnitude F_0 is applied to the string as shown. The string is unstretchable, has negligible mass, and pulls without slipping. The pulley has radius R and moment of inertia I with respect to the axis of rotation. There is a friction force at the axle which exerts a torque of constant magnitude τ_0 opposing the rotation.



$F_x = m a_x$

$\vec{\tau}_{ext} = I\alpha$ (rhr)

a) In the box below write the system of equations that could be solved to find the acceleration of the block.

$$-m_1 g \sin\theta + T_1 = m_1 a_x$$
$$F_0 R - T_1 R - \tau_0 = I\alpha$$
$$a_x = R\alpha$$

b) Find the acceleration of the block.

$$-m_1 g \sin\theta + T_1 = m_1 a_x$$

$$T_1 = \frac{-\tau_0 + F_0 R - I\frac{a_x}{R}}{R}$$

$$-m_1 g \sin\theta - \frac{\tau_0}{R} + F_0 - I\frac{a_x}{R^2} = m_1 a_x$$

$$a_x = \frac{F_0 - m_1 g \sin\theta - \frac{\tau_0}{R}}{\left(m_1 + \frac{I}{R^2}\right)}$$