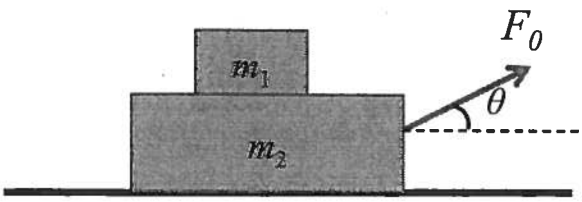
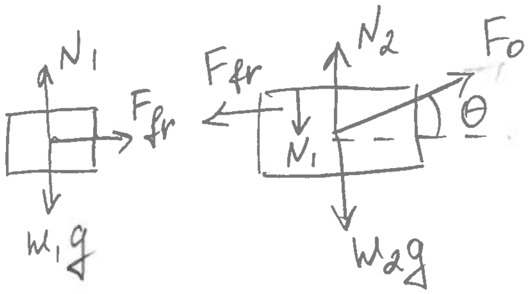
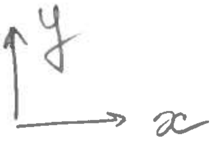
Problem 1: (15 points)

Two blocks of masses m_1 and m_2 are moving together on a frictionless surface. A force F_0 is applied at a angle θ as shown. There is friction between the blocks. Draw the free body diagram for both blocks. Find the force that m_2 exerts on m_1.







$$F_x = m a_x$$
$$F_y = m a_y$$

$$\begin{cases} F_{fr} = m_1 a_x \\ N_1 - m_1 g = 0 \\ F_0 \cos\theta - F_{fr} = m_2 a_x \\ N_2 - N_1 - m_2 g + F_0 \sin\theta = 0 \end{cases}$$

$$N_1 = m_1 g$$
$$a_x = \frac{F_{fr}}{m_1}$$

$$F_0 \cos\theta - F_{fr} = \frac{m_2}{m_1} F_{fr}$$

$$F_{fr} = \frac{F_0 \cos\theta}{1 + \frac{m_2}{m_1}}$$

Answer:

$$\vec{F} = \frac{F_0 \cos\theta}{1 + \frac{m_2}{m_1}} \vec{i} + m_1 g \vec{j}$$

Problem 2: (12 points)

In a nuclear collision an incoming proton has initial velocity of magnitude v_0 (let's call it $+x$ direction). It collides with another proton, initially at rest. After the collision one proton goes off at angle θ to the x axis. The second proton goes off at unknown angle Φ. If the collision is perfectly *elastic*, in the box below write the system of equations that could be solved to find the velocities of both protons after the collision and angle Φ. Do not solve.

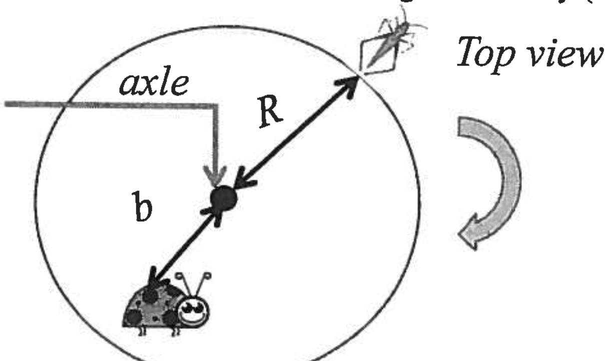
$$\cancel{m} v_0 = \cancel{m} v_1 \cos\theta + \cancel{m} v_2 \cos\Phi$$

$$0 = \cancel{m} v_1 \sin\theta - \cancel{m} v_2 \sin\Phi$$

$$\frac{\cancel{m} v_0^2}{2} = \frac{\cancel{m} v_1^2}{2} + \frac{\cancel{m} v_2^2}{2}$$

Problem 3 : (15 points)

A platform of radius R can rotate, without friction, about a vertical axle through its center with a moment of inertia, I_p. A small bug of mass m_b is placed on the platform at a distance b from the center. A tiny grasshopper of mass m_g is placed at the edge of the platform at distance R from the axle. The system is set spinning with angular velocity ω_0. (Clockwise as viewed from above). At $t = 0$ the grasshopper jumps off the platform along the radius with velocity v_g, and the bug starts running with velocity v_b with respect to the ground in a circle of radius b in the opposite direction to the platform's rotation. What is the new angular velocity (magnitude and direction) of the system?



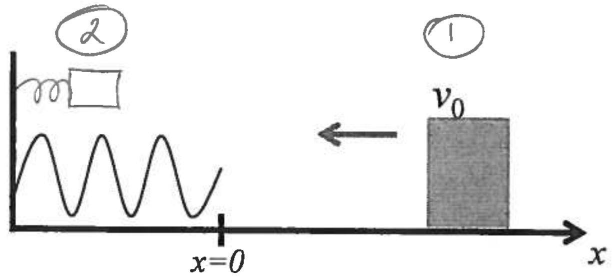
$\vec{L}$ (initial) = $\vec{L}$ (final)

$$I_p \omega_0 + m_b b^2 \omega_0 + m_g R^2 \omega_0 = I_p \omega - m_b v_b b$$

$$\omega = \frac{(I_p + m_b b^2 + m_g R^2)\omega_0 + m_b v_b b}{I_p} \quad \text{CW}$$

Problem 4: (15 points)

A block of mass m moves to the left with velocity v_0 on a frictionless surface. It runs into a spring and compresses it. The force of the spring is $F_x = -(\alpha x + \beta x^5)$. Prove that the force of the spring is a conservative force. Find the maximum compression of the spring, x_s. Obtain the equation, but do not solve it.



$$F_x = -(\alpha x + \beta x^5)$$

$$F_x = -\frac{dU}{dx}; \quad U = -\int F_x\,dx = -\int -(\alpha x + \beta x^5)\,dx =$$

$$= \frac{\alpha x^2}{2} + \frac{\beta x^6}{6} + \text{Const}$$

$$U_1 + KE_1 = U_2 + KE_2$$

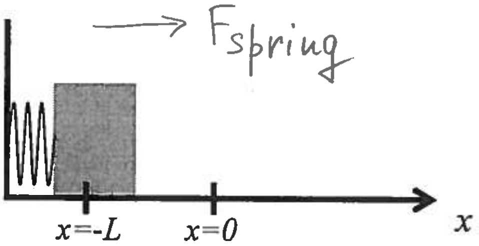
$$\boxed{0 + \frac{m v_0^2}{2} = \frac{\alpha x_s^2}{2} + \frac{\beta x_s^6}{6}}$$

$$\text{Or } W^{net} = \int_0^{x_s} F_x\,dx = \int_0^{x_s} -(\alpha x + \beta x^5)\,dx = 0 - \frac{m v_0^2}{2}$$

$$\boxed{-\frac{\alpha x_s^2}{2} - \frac{\beta x_s^6}{6} = -\frac{m v_0^2}{2}}$$

Problem 5: (15 points)

A block of mass m is attached to a spring, spring constant k. The spring is compressed by amount of L from the point at which the spring is unstretched. The block is on a frictionless table and released from rest. a) Find the position of the block at any time moment. b) How long will it take for the block to return to $x = 0$?



$$x(t=0) = -L$$
$$v(t=0) = 0$$

$$F_x = ma_x$$

$$-kx = ma_x = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$x(t=0) = \boxed{A = -L}$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$v(t=0) = B\omega = 0 \Rightarrow \boxed{B = 0}$$

$$x(t) = A\cos(\omega t)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t)$$

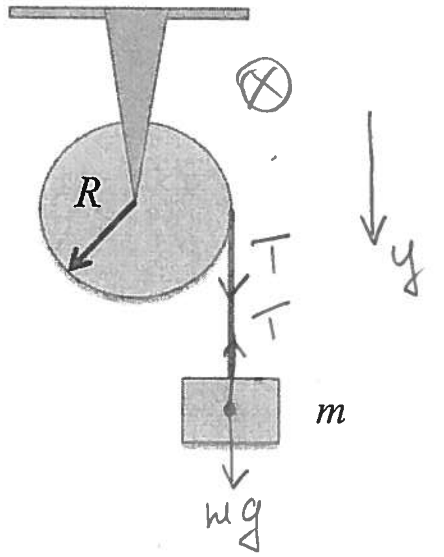
$$-\cancel{A}\omega^2\cancel{\cos(\omega t)} + \frac{k}{m}\cancel{A\cos(\omega t)} = 0$$

$$\omega^2 = \frac{k}{m}; \quad \omega = \sqrt{\frac{k}{m}}$$

$$\boxed{x(t) = -L\cos\left(\sqrt{\frac{k}{m}}\,t\right)} \qquad t^* = \frac{T}{4} = \frac{2\pi}{4\omega} = \boxed{\frac{\pi\sqrt{m}}{2\sqrt{k}}}$$

Problem 6: (13 points)

A pulley has mass M_p, radius R and moment of inertia about an axis through its center I_p. A block of mass m is hung from the rope that moves without slipping on the pulley. a) In the box below write the system of equations that can be solved to find the acceleration of the block. The problem will not be graded without free body diagrams and coordinate system.



$$F_y = ma_y$$

$$\vec{\tau}_{ext} = I\alpha \ (rhr)$$

$$a_y = R\alpha$$

$$\boxed{\begin{aligned} mg - T &= ma_y \\ RT &= I_p \alpha \\ a_y &= R\alpha \end{aligned}}$$

b) Find the acceleration of the block. Find the angular velocity of pulley's rotation as a function of time if the block is released from rest.

$$RT = I_p \frac{a_y}{R}$$

$$T = I_p \frac{a_y}{R^2}$$

$$mg - I_p \frac{a_y}{R^2} = ma_y$$

$$\boxed{a_y = \frac{mg}{m + \frac{I_p}{R^2}}}$$

$$\alpha = \frac{a_y}{R} = \frac{mg}{R\left(m + \frac{I_p}{R^2}\right)}$$

$$\omega = \int \alpha \, dt = \boxed{\frac{mg}{R\left(m + \frac{I_p}{R^2}\right)} t}$$

Problem 7: (15 points)

A car of mass m travels on a flat, circular curve, radius R. The coefficient of friction between the car and the road is μ. Suppose at $t = 0$ the car had velocity of magnitude v_0 and was given a constant angular acceleration c_1 about the center of the curve. Find the radial and tangential components of the force of friction. At what time will the car begin to slip off the road? Obtain an equation in terms of R, g, v_0, μ, m, and c_1. Do not solve it.

$$F_r = ma_r = m\left(\frac{d^2r}{dt^2} - r\omega^2\right)$$

$$F_\theta = ma_\theta = m\left(2\frac{dr}{dt}\omega + r\alpha\right)$$

$$r = \text{Const}$$

$$\begin{cases} F_r = -mR\omega^2 \\ F_\theta = mR\alpha \end{cases}$$

$$\omega(t) = \int \alpha(t)\,dt = \int c_1\,dt = c_1 t + \text{Const} = c_1 t + \frac{v_0}{R}$$

$$\begin{cases} F_r = -mR\left(c_1 t + \frac{v_0}{R}\right)^2 \\ F_\theta = mRc_1 \end{cases}$$

$$F_{fr} = \sqrt{F_r^2 + F_\theta^2} = \mu N$$

$$F_y = ma_y; \quad N - mg = 0$$

$$\boxed{\sqrt{m^2R^2\left(c_1 t + \frac{v_0}{R}\right)^4 + m^2R^2c_1^2} = \mu mg}$$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{total} \cdot d\vec{r} = \frac{mV_{final}^2}{2} - \frac{mV_{initial}^2}{2}$$

$$W^{non-conservative} = [U(\vec{r}_2) + \frac{mV_2^2}{2}] - [U(\vec{r}_1) + \frac{mV_1^2}{2}]$$

$$a_r = \frac{d^2r}{dt^2} - r\omega^2; \quad a_\theta = 2\frac{dr}{dt}\omega + r\alpha$$

$$\omega = \frac{d\theta}{dt}; \quad \alpha = \frac{d\omega}{dt}$$

$$V_r = \frac{dr}{dt}; \quad V_\theta = r\frac{d\theta}{dt} = r\omega$$

$$\frac{d\vec{L}_{tot}}{dt} = \vec{\tau}_{ext}; \quad \vec{L} = \vec{r} \times \vec{p}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{L} = I\omega\,(rhr); \quad I = \sum_i m_i r_i^2$$

$$\vec{\tau}_{ext} = \frac{d\vec{L}_{tot}}{dt}$$

$$\vec{\tau}_{ext} = I\alpha\,(rhr)(\text{constant R})$$

$$F_x = -\frac{\partial U}{\partial x}; \quad F_y = -\frac{\partial U}{\partial y}$$

$$\frac{d\vec{p}}{dt} = \vec{F}_{ext}$$

$$v_x(t) = \frac{dx(t)}{dt}$$

$$a_x(t) = \frac{dv_x(t)}{dt}$$

If a=a_c=$Const$:

$$x(t) = \frac{1}{2}a_{cx}t^2 + v_x(0)t + x(0)$$

$$v_x(t) = a_{cx}t + v_x(0)$$

$$v_x^2(t_2) - v_x^2(t_1) = 2a_{cx}(x(t_2) - x(t_1))$$