Problem 1: (15 points)

An object with mass m initially at rest is acted on by a single force $\vec{F} = k_1\vec{i} + k_2t^2\vec{j}$, where k_1 and k_2 are known constants. Calculate the velocity of the object as a function of time.

$$F_x = k_1 = ma_x$$

$$F_y = k_2t^2 = ma_y$$

$$a_x = \frac{k_1}{m}; \quad a_y = \frac{k_2t^2}{m}$$

$$v_x = \int a_x(t)dt = \int \frac{k_1}{m}dt = \frac{k_1}{m}t + v_x(0)$$

$$v_x(0) = 0$$

$$v_x = \frac{k_1}{m}t$$

$$v_y = \int a_y(t)dt = \int \frac{k_2t^2}{m}dt =$$

$$= \frac{k_2}{m}\frac{t^3}{3} + v_y(0)$$

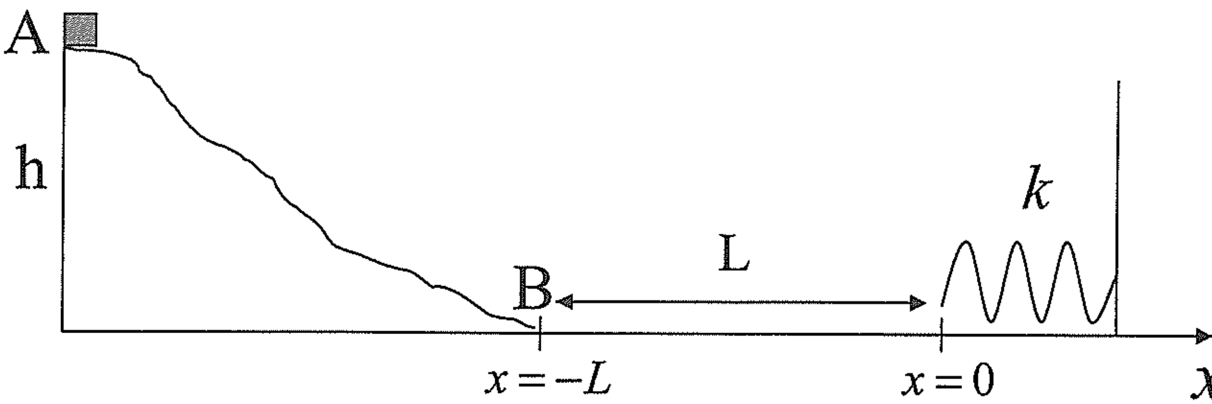
$$v_y(0) = 0$$

$$v_y = \frac{k_2}{m}\frac{t^3}{3}$$

$$\vec{v}(t) = \frac{k}{m}t\vec{i} + \frac{k_2}{m}\frac{t^3}{3}\vec{j}$$

Problem 2: (20 points)

A block of mass m slides down a snow-covered hill, height h, starting at point A from rest. There is no friction between points A and B, but there is friction on the level ground at the bottom of the hill, between B and the wall. The coefficient of friction is $\mu = \mu_0(1+\alpha x)$, where μ_0 and α are known constants. After entering the rough horizontal region, the stone travels distance L and then runs into a very light spring with constant k.



a) Find the velocity of the block at point B. $U(A) + \frac{mv^2}{2}(A) = U(B) + \frac{mv^2}{2}(B)$

$mgh = \frac{mv_B^2}{2}$ $\quad \boxed{v_B = \sqrt{2gh}}$

b) How far will the stone compress the spring if the origin is at the unstretched position? Indicate the method you use to solve the problem. Write the correct equation. Do not solve it.

$W_{non\text{-}conservative} = U_{final} + KE_{final} - U_{initial} - KE_{initial}$

$W_{friction} = \int_{-L}^{x_s} -\mu_0(1+\alpha x)mg\,dx = \left(-\mu_0 x - \mu_0 \alpha \frac{x^2}{2}\right)mg\Big|_{-L}^{x_s} =$

$= -\mu_0 mg\left[(x_s + L) + \alpha\left(\frac{x_s^2}{2} - \frac{L^2}{2}\right)\right]$

$U_{final} = \frac{kx_s^2}{2}$; $U_{initial} = mgh$; $KE_{final} = 0$; $KE_{initial} = 0$

$\boxed{-\mu_0 mg\left[(x_s + L) + \alpha\left(\frac{x_s^2}{2} - \frac{L^2}{2}\right)\right] = \frac{kx_s^2}{2} - mgh}$

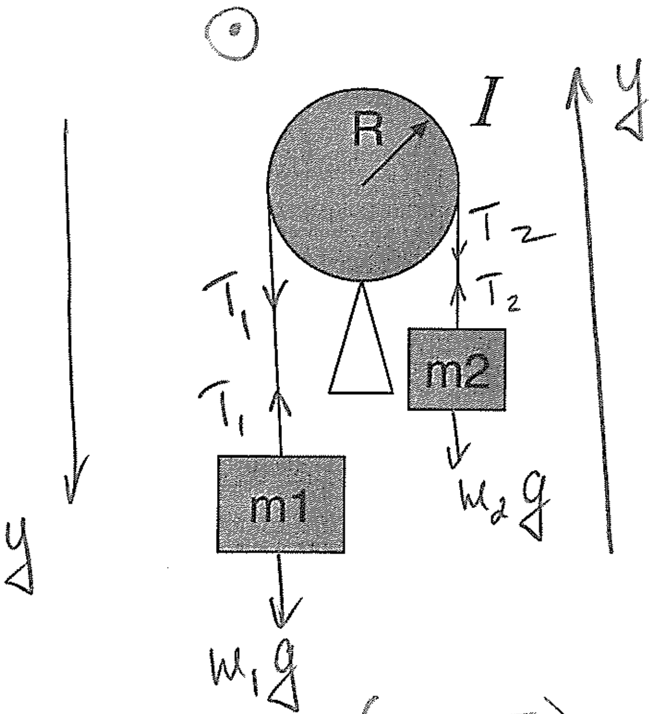
Or: Work Energy Theorem: $W^{spring} = \int_0^{x_s} -kx\,dx = -\frac{kx_s^2}{2}$

$v_{initial} = 0$; $v_{final} = 0$ $\quad W_{gravity} = mgh$; $W^N = 0$.

Or: $v_{initial} = v_B$ $\quad W^{net} = W^{friction} + W^{spring}$

Problem 3: (20 points)

A pulley with radius R and moment of inertia about its central axis, I, is mounted on frictionless bearings. A massless cord is wrapped around the axle with two masses attached to it. Assuming the cord does not slip, find the acceleration of mass m_1.



$$F_y = m a_y \; ; \quad \tau_{ext} = I\alpha \; (\text{rhr})$$

$$\begin{cases} T_2 - m_2 g = m_2 a_2 & (1) \\ m_1 g - T_1 = m_1 a_1 & (2) \\ a_1 = a_2 & (3) \\ T_1 R - T_2 R = I\alpha & (4) \\ a_1 = R\alpha & (5) \end{cases}$$

$$(T_1 - T_2)R = I \frac{a_1}{R}$$

$$T_1 - T_2 = I \frac{a_1}{R^2}$$

$$(1) + (2): \quad m_1 g - (T_1 - T_2) - m_2 g = (m_1 + m_2) a_1$$

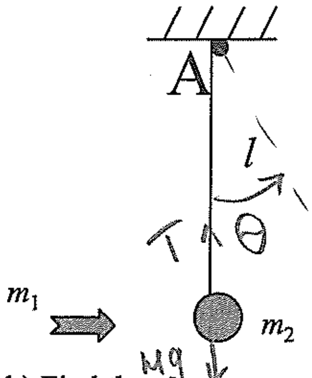
$$(m_1 - m_2) g - I \frac{a_1}{R^2} = (m_1 + m_2) a_1$$

$$\boxed{a_1 = \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{I}{R^2}}}$$

Problem 4: (20 points)

A dart, mass m_1, moves horizontally with a velocity of magnitude v_0. It strikes a simple pendulum, a small object, mass m_2, suspended from a massless string of length l. Assume the collision of the dart and the small object takes place so quickly that the string remains vertical during the collision.

a) Find the velocity of the pendulum and embedded dart right after the collision.



$$P_x(\text{before}) = P_x(\text{after})$$

$$m_1 v_0 = (m_1 + m_2)\,u$$

$$\boxed{u = \frac{m_1}{m_1 + m_2} v_0}$$

b) Find the angular momentum of the pendulum and embedded dart about the point A right after the collision.

$\vec{L} = \vec{r} \times \vec{p}$; $r = l$; $p = (m_1 + m_2)u$

$$\boxed{L = (m_1 + m_2)\,u\,l = m_1 v_0 l} \odot$$

c) Find the dependence $\theta(t)$ if the displacement is small ($\sin\theta \approx \theta$).
How long will it take for the pendulum to return to the equilibrium.

$\vec{\tau}_{ext} = I\alpha$ (hhr); $\tau \otimes \to R$, $(m_1 + m_2)g$

$$-(m_1 + m_2) g l \sin\theta = (m_1 + m_2) l^2 \alpha$$

$\alpha = \frac{d^2\theta}{dt^2}$; $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\theta(t) = A\cos\omega t + B\sin\omega t$$

$\theta(0) = A = 0$; $\frac{d\theta}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$

$\frac{d\theta}{dt}(0) = B\omega = \frac{u}{l}$; $B = \frac{m_1}{m_1 + m_2}\frac{v_0}{\omega l}$

$$\boxed{\theta(t) = \frac{m_1}{m_1 + m_2}\frac{v_0}{\omega l}\sin\omega t}$$

$\omega t = \pi$

$$\boxed{t = \frac{\pi}{\omega} = \pi\sqrt{\frac{l}{g}}}$$

Problem 5: (20 points)

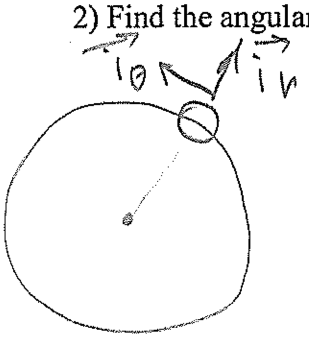
An object of mass m circles the earth and is **attracted** to it with a force whose magnitude is given by

$$|\vec{F}| = G\frac{m_E m}{r^2}$$

1) Show that this force is conservative.

$$U = -\int F_r\, dr = -\int -G\frac{m_E m}{r^2}\, dr = -\frac{G m_E m}{r} + C$$

2) Find the angular velocity if the radius of the orbit is R.



$$F_r = m a_r$$

$$-G\frac{m_E m}{R^2} = -m R \omega^2$$

$$\omega = \sqrt{\frac{G m_E}{R^3}}$$

3) Find the work done by this force if the radius of the orbit is changed from R to 1.5R.

$$W = \int_R^{1.5R} -G\frac{m_E m}{r^2}\, dr = G\frac{m_E m}{r}\Bigg|_R^{1.5R} = G m_E m\left(\frac{1}{1.5R} - \frac{1}{R}\right)$$

or $$W = -\left[U(1.5R) - U(R)\right] = G m_E m\left(\frac{1}{1.5R} - \frac{1}{R}\right)$$

$$\boxed{W = -\frac{G m_E m}{3R}}$$

Problem 6: (10 points)

An object with mass m is acted by two forces, both in x direction: $F_1 = -\alpha x$ and $F_2 = -\beta x$. Find the position of this object as a function of time, $x(t)$, if initially the object was at rest at distance x_0 from the origin.

$$F_x = m a_x$$

$$-\alpha x - \beta x = m \frac{d^2 x}{dt^2}$$

$$\frac{d^2 x}{dt^2} + \frac{\alpha + \beta}{m} x = 0.$$

$$x(t) = A \cos \omega t + B \sin \omega t$$

$$\omega = \sqrt{\frac{\alpha + \beta}{m}}$$

$$x(0) = x_0 = A$$

$$\frac{dx}{dt} = -A \omega \sin \omega t + B \omega \cos \omega t$$

$$\frac{dx}{dt}(t=0) = B\omega = 0$$

$$\boxed{x(t) = x_0 \cos \omega t}$$